EXTENSIONS OF BANACH ALGEBRAS

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1. Introduction. We are concerned with propositions of four types (1.1–1.4) about a commutative Banach algebra A and its various commutative Banach algebra extensions B.

1.1 TPr. *If* $\{B_i : i \in I\}$ *is a family of extensions of* A, *then there is an extension* B *of* A *and topological isomorphism* $\{f_i : i \in I\}$ *where* $f_i(B_i) \subset B$ *and* $f_i(a) = a$ *for* $a \in A$.

Let us call [*normally*] *solvable over* A a system Σ of polynomials over A (or more generally, multiple power series elements) such that there is an extension B of A in which there is a system of elements [whose norms do not exceed 1 and] whose substitution into Σ reduces each member equal to 0.

1.2 Sol. *Let* $\{\Sigma_i : i \in I\}$ *be a family of solvable systems such that no indeterminate occurs in more than one system. Then* $\Sigma = \bigcup \Sigma_i$ *is solvable.*

A system $\mathscr{J}$ of ideals is *removable* if in some extension, each ideal J of $\mathscr{J}$ generates the ideal (1).

1.3 RId. *Let* $\{J_i : i \in I\}$ *be a family of removable ideals. Then it is a removable system.*

An element $c \in A$ is called [*normally*] *subregular* if it has an inverse [of norm $\leqq 1$] in some extension.

1.4 Inv. *Let* $\{c_i : i \in I\}$ *be a family of subregular elements. Then, in some extension, each* c_i *has an inverse.*

Our findings on such propositions is that **TPr** is false, and that **Inv** is true if I is finite, but false if a natural norm restriction is brought in. By the *finite* form of 1.1–1.4 we mean that in which I is finite. By the *normal* form we mean the statements obtained if in (1.1) the f_i are required to be isometries, if 'solvable' in (1.2) is replaced by 'normally solvable', and 'subregular' in (1.4) by 'normally subregular'.

This gives four forms of propostions of each type:

(1.5)	*normal*	(*no qualification*)
	finite normal	*finite* .

For each type (1.1–1.4), there are rather obvious implications in (1.5), namely to the right, and downward. (To see this, one need only observe that c is subregular if and only if λc is normally subregular for some $\lambda \in \boldsymbol{C}$, etc.). For each form (1.5) there are implications among the

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