LATTICES WHOSE CONGRUENCES FORM A BOOLEAN ALGEBRA

PETER CRAWLEY

1. Introduction. The structure of a lattice L, in particular its representation as direct or subdirect unions, depends heavily on the structure of $\Theta(L)$, its lattice of congruence relations. Thus a natural problem which arises is that of determining those lattices whose congruence lattices have certain specified properties. Two problems of this general type are considered here.

The first problem involves the sequence of iterated congruence lattices. If D is a distributive lattice, then as is well known, D is isomorphic with a sublattice of $\Theta(D)$. Hence the iterations of $\Theta(D)$ form an ascending chain, $D \leq \Theta(D) \leq \Theta(\Theta(D)) \leq \cdots$. We shall show that this chain terminates if and only if D is finite, thus answering a question of Morgan Ward. This result is a consequence of the theorem stating that if L is an arbitrary lattice, then $L \cong \Theta(L)$ if and only if L is a finite boolean algebra. The proof of the latter statement rests on the following embedding theorem: if D is a distributive lattice of infinite cardinality α, then every distributive lattice of cardinality at most 2_{α} can be embedded in $\Theta(\Theta(\Theta(\Theta(D))))$.

The second problem considered is that of characterizing those lattices L for which $\Theta(L)$ is a boolean algebra.[1] This problem has been solved previously by Tanaka [5] and by Grätzer and Schmidt [3], however, neither of these solutions is given entirely in terms of the intrinsic structure of the lattice L. Here such an "intrinsic" characterization is obtained, which has the form of a finiteness condition on the lattice. Several applications of this result are given, which illustrate the finiteness nature of the condition of complementation of $\Theta(L)$.

The main tool used in this investigation is the *compactly generated* property of $\Theta(L)$ (see § 2). This property is a generalization of the ascending chain condition, and many important consequences of the chain condition also hold in lattices which are compactly generated.

2. Preliminaries. Throughout this note the familiar notation and terminology is used. The unit and null elements of a lattice L will be denoted by u and z respectively. A quotient a/b is defined by $a/b = \{x \in L \mid b \leq x \leq a\}$. We say a/b is prime if a covers b. Each of the quotients $a/a \cap b$ and $a \cup b/b$ is said to be a transpose of the other. And a quotient c/d is weakly projective into a/b if there exists a finite

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[1] This is listed as Problem 72 in [**1**].