NORMAL EXTENSIONS OF FORMALLY NORMAL OPERATORS

EARL A. CODDINGTON

1. Introduction. Let $\mathfrak{H}$ be a Hilbert space. If T is any operator in $\mathfrak{H}$ its domain will be denoted by $\mathfrak{D}(T)$, its null space by $\mathfrak{N}(T)$. A *formally normal* operator N in $\mathfrak{H}$ is a densely defined closed operator such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$, and $\|Nf\| = \|N^*f\|$ for all $f \in \mathfrak{D}(N)$. Intimately associated with such an N is the operator $\bar{N}$ which is the restriction of N^* to $\mathfrak{D}(N)$. The operator N is formally normal if and only if $\bar{N}$ is. A *normal operator* N in $\mathfrak{H}$ is a formally normal operator for which $\mathfrak{D}(N) = \mathfrak{D}(N^*)$; in this case $\bar{N} = N^*$. A densely defined closed operator N is normal if and only if $N^*N = NN^*$.[1]

Let N be formally normal in $\mathfrak{H}$. Since $\bar{N} \subset N^*$ we have $N \subset \bar{N}^*$, where $\bar{N}^* = (\bar{N})^*$. Thus we see that a closed symmetric operator is a formally normal operator such that $N = \bar{N}$, and a self-adjoint operator is a normal operator such that $N = \bar{N}$ $(= N^*)$. If a closed symmetric operator has a normal extension in $\mathfrak{H}$, this extension is self-adjoint. It is known that a closed symmetric operator may not have a self-adjoint extension in $\mathfrak{H}$. Necessary and sufficient conditions for such extensions were given by von Neumann.[2] However, until recently, conditions under which a formally normal operator N can be extended to a normal one in $\mathfrak{H}$ were known only for certain special cases.[3,4] Kilpi[5] considered the problem in terms of the real and imaginary parts of N. It is the purpose of this note to characterize the normal extensions of N in a manner similar to the von Neumann solution for the symmetric case.

If N_1 is a normal extension of a formally normal operator N in $\mathfrak{H}$, then it is easy to see that $N \subset N_1 \subset \bar{N}^*$, and $\bar{N} \subset N_1^* \subset N^*$. In Theorem 1 we describe $\mathfrak{D}(\bar{N}^*)$ and $\mathfrak{D}(N^*)$ for any two operators $N, \bar{N}$ satisfying $N \subset \bar{N}^*$, $\bar{N} \subset N^*$. With the aid of this result a characterization of the normal extensions N_1 of a formally normal N in $\mathfrak{H}$ is given in Theorem 2. It is indicated in Theorem 3 how the domains of normal extensions

Received January 13, 1960. This work was supported in part by the National Science Foundation and the Office of Naval Research.

1 See, e.g., B. v. Sz. Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Ergeb. Math., **5** (1942), 33.

2 Ibid; p. 39.

3 Y. Kilpi, "Über lineare normale Transformationen im Hilbertschen Raum", Annales Academiae Scientiarum Fennicae, Series A-I, No. **154** (1953).

4 R. H. Davis, "Singular normal differential operators", Technical Report No. 10, Department of Mathematics, University of California, Berkeley, Calif., (1955).

5 Y. Kilpi, "Über das komplexe Momentenproblem", Annales Academiae Scientiarum Fennicae, Series A-I, No. **236** (1957).