CONTINUITY AND CONVEXITY OF PROJECTIONS AND BARYCENTRIC COORDINATES IN CONVEX POLYHEDRA

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If $s_0, \cdots, s_n$ are linearly independent points of real n-dimensional Euclidean space R^n then each point x of their convex hull S has a (unique) representation $x = \sum_{i=0}^n \lambda_i(x)s_i$ with $\lambda_i(x) \geqq 0$ $(i = 0, \cdots, n)$ and $\sum_{i=0}^n \lambda_i(x) = 1$, and the barycentric coordinates $\lambda_0, \cdots, \lambda_n$ are continuous convex functions on S (cf. [3, p. 288]). We shall show in this paper that given any finite set $s_0, \cdots, s_m$ of points of R^n we can assign barycentric coordinates $\lambda_0, \cdots, \lambda_m$ to their convex hull S in such a way that each coordinate is continuous on S and that one prescribed coordinate (λ_0 say) is convex on S (Theorem 2); the author does not know whether it is always possible to make all the coordinates convex simultaneously (cf. Example 3). In proving Theorem 2 we shall use certain "projections" which we now define; these projections are in general distinct from those of [1, p. 614] and [2, p. 12]. Given two distinct points s_0 and s of R^n, let s_0s be the open half-line consisting of all points $s_0 + \lambda(s - s_0)$ with $\lambda > 0$; given a point s_0 of R^n and a closed subset S of R^n such that $s_0 \notin S$, let $C(s_0, S)$ be the "cone" formed by the union of all open half-lines s_0s with s in S; and given a point x in such a cone $C(s_0, S)$, let $\pi(x)$ be the (unique) point of $s_0x \cap S$ which is closest to s_0. Then we shall call the function π the "projection of $C(s_0, S)$ on S." Our proof of Theorem 2 depends on the fact that if S is a convex polyhedron then π is continuous (Theorem 1). This result may appear to be obvious, but it is not immediately obvious how a formal proof should be given; moreover, as we shall show (Examples 1 and 2), the conclusion need not remain true for polyhedra S which are not convex or for convex sets S which are not polyhedra. The author is indebted to the referee for improvements to Lemma 3, Example 1, and Example 2, and for the remark at the end of § 1.

1. Projections. For any subset A of R^n we shall denote by $H(A)$ the convex hull of A and by $L(A)$ the affine subspace of R^n spanned by A (cf. [2, pp. 21, 15]). If $A = \{s_1, \cdots, s_p\}$ we shall write $H(A) = H(s_1, \cdots, s_p)$ and $L(A) = L(s_1, \cdots, s_p)$. Given two points x and y of R^n we shall denote by (x, y) the inner product of x and y and by $|x - y|$ the Euclidean distance $\sqrt{(x - y, x - y)}$ between x and y.

LEMMA 1. *Let s_0 be a point of R^n, let S be a closed convex subset of R^n such that $s_0 \notin S$, and let π be the projection of $C(s_0, S)$ on S. Suppose that points $x, s_1, \cdots, s_p$ of S and real numbers $\lambda_1, \cdots, \lambda_p$ are*

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