A NOTE ON GENERALIZATIONS OF SHANNON-MCMILLAN THEOREM

SHU-TEH C. MOY

1. Introduction. This paper is a sequel to an earlier paper [6]. All notations in [6] remain in force. As in [6] we shall consider tw probability measures μ, ν an the infinite product σ-algebra of subsets of the infinite product space $\Omega = \pi X$. ν is assumed to be *stationary* and μ to be *Markovian with stationary transition probabilities*. Extensions to K-Markovian μ are immediate. $\nu_{m,n}$, the contraction of ν to $\mathscr{F}_{m,n}$, is assumed to be *absolutely continuous* with respect to $\mu_{m,n}$, the contraction of μ to $\mathscr{F}_{m,n}$, and $f_{m,n}$ is the Radon-Nikodym derivative. In [6] the following theorem is proved. If $\int \log f_{0,0}\, d\nu < \infty$ and if there is a number M such that

$$(1) \qquad \int (\log f_{0,n} - \log f_{0,n-1}) d\nu \leqq M \text{ for } n = 1, 2, \cdots$$

then $\{n^{-1} \log f_{0,n}\}$ converges in $L_1(\nu)$. (1) is also a necessary condition for the $L_1(\nu)$ convergence of $\{n^{-1} \log f_{0,n}\}$. We consider this theorem as a generalization of the Shannon-McMillan theorem of information theory. In the setting of [6] the Shannon-McMillan theorem may be stated as follows. Let X be a finite set of K points. Let ν be any stationary probability measure of $\mathscr{F}$, and μ the equally distributed independent measure on $\mathscr{F}$. Then $\{n^{-1} \log f_{0,n}\}$ converges in $L_1(\nu)$. In fact, the $P(x_0, x_1, \cdots, x_n)$ of Shannon-McMillan is equal to $K^{(n+1)} f_{0,n}$. The convergence with probability one of $\{n^{-1} \log P(x_0, \cdots, x_n)\}$ for a finite set X was proved by L. Breiman [1] [2]. K.L. Chung then extended Breiman's result to a countable set X. [3]. In this paper we shall prove that the convergence with ν-probability one of $\{n^{-1} \log f_{0,n}\}$ follows from the following condition.

$$(2) \qquad \int \frac{f_{0,n}}{f_{0,n-1}} d\nu \leqq L,\ n = 1, 2, \cdots .$$

(2) is a stronger condition than (1) since by Jensen's inequality

$$\log \int \frac{f_{0,n}}{f_{0,n-1}} d\nu \geqq \int \log \frac{f_{0,n}}{f_{0,n-1}} d\nu .$$

An application to the case of countable X is also discussed.

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