EIGENVALUES OF SUMS OF HERMITIAN MATRICES

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Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be arbitrary nonincreasing sequences of real numbers. We consider the question: for which nonincreasing sequences $\gamma = (\gamma_1, \cdots, \gamma_n)$ do there exist Hermitian matrices A and B such that A, B and $A + B$ have α, β and γ respectively as their sequences of eigenvalues. Necessary conditions have been obtained by several authors including Weyl [4], Lidskii [3], Wielandt [5], and Amir-Moez [1]. Besides the obvious condition

$$(1) \qquad \gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n \, ,$$

these conditions are linear inequalities of the form

$$(2) \qquad \gamma_{k_1} + \cdots + \gamma_{k_r} \leqq \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots \beta_{j_r} \, ,$$

where i, j and k are increasing sequences of integers. As far as I know all other known necessary conditions are consequences of these inequalities. It is therefore natural to conjecture that the set E of all possible γ forms a convex subset of the hyperplane (1). The set E has hitherto not been determined except in the simple cases $n = 1, 2,$ and will not be determined in general here.

In § 2, which is independent of § 1, we are going to give a method of finding conditions of the form (2) which will yield many new ones. We shall find all possible inequalities (2) for $r = 1, 2,$ and arbitrary n, and establish a large class of such inequalities for $r = 3$. In § 1, we use Lidskii's method to find a necessary condition on the boundary points of a subset E' of E. These results are used in § 3 to determine the set E for $n = 3, 4$. In addition a conjecture is given for E in general.

If x is a sequence, x_p denotes the pth component of x. If A is a matrix, A^* and A' denote the conjugate transpose and transpose of A. If i is a sequence of integers such that $1 \leqq i_1 < \cdots < i_r \leqq n$, by the complement of i with respect to n we mean the sequence obtained by deleting the terms of i from the sequence $1, 2, \cdots, n$. If α is a sequence of numbers, diag $(\alpha_1, \cdots, \alpha_n)$ denotes the diagonal matrix with diagonal α. If M and N are matrices, diag (M, N) denotes the direct sum matrix

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} .$$

The inner product of the vectors x and y is written (x, y). I_r is the

Received January 16, 1961. Part of this research was supported by a National Science Foundation grant.