ON THE NUMBER OF PURE SUBGROUPS

PAUL HILL

A problem due to Fuchs [3] is to determine the cardinality of the set $\mathscr{P}$ of all pure subgroups of an abelian group. Boyer has already given a solution for nondenumerable groups G [1]; he showed that $|\mathscr{P}| = 2^{|G|}$ if $|G| > \aleph_0$, where $|A|$ denotes the cardinality of a set A. Our purpose is to complement the results of [1] by determining those groups for which $|\mathscr{P}|$ is finite, $\aleph_0$, and $c = 2^{\aleph_0}$. In the following, group will mean abelian group.

LEMMA 1. *If G is a torsion group with $|G| \leqq \aleph_0$, then $|\mathscr{P}| = c$ unless*

$$G = p_1^\infty \oplus p_2^\infty \oplus \cdots \oplus p_n^\infty \oplus B\,, \tag{1}$$

a direct sum of (at most) a finite number of groups of type p^∞ and a finite group, where $p_i \neq p_j$ if $i \neq j$. If G is of the form (1), *then $|\mathscr{P}|$ is finite.*

Proof. The latter statements is clear, and if none of the following hold

(i) G decomposes into an infinite number of summands

(ii) G contains $p^\infty \oplus p^\infty$ for some prime p

(iii) $|B| = \aleph_0$, where B is the reduced part of G,

then G is of the form (1). Moreover, if (i) holds, then obviously $|\mathscr{P}| = c$. Every automorphism of p^∞ determines a pure subgroup of $p^\infty \oplus p^\infty$, and distinct automorphisms correspond to distinct subgroups. Since $|A(p^\infty) = \text{automorphism group}| = c$, it follows that $p^\infty \oplus p^\infty$ has c pure subgroups. Thus if (ii) holds, $|\mathscr{P}| = c$ since $p^\infty \oplus p^\infty$ is a direct summand of G. Finally, if (iii) holds and if (i) does not, then the following argument shows that $|\mathscr{P}| = c$. We may write[1] $B = C_1 \oplus B_1 = C_1 \oplus C_2 \oplus B_2$, and continuing in this way define an infinite sequence C_n of cyclic groups such that no C_i is contained in the direct sum of any of the others. The direct sum of any subcollection of these cyclic groups is a pure subgroup of B and, therefore, of G.

An interesting corollary is noted: there is no torsion group with exactly $\aleph_0$ pure subgroups.

LEMMA 2. *If $G = F \oplus B$ is the direct sum of a torsion free group*

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[1] This is precisely the proof of Boyer that such a group has c subgroups [2].