AN INEQUALITY FOR CLOSED SPACE CURVES

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1. Among a number of interesting results in a paper of I. Fáry (see [2]) appears the following. Let C be a rectifiable closed curve of length $L(C)$ and total curvature $\kappa(C)$ enclosed by a sphere S of radius r in Euclidean 3-space. Then

$$L(C) \leqq \frac{4}{\pi} r\kappa(C) . \tag{1}$$

The proof of (1) rests upon the corresponding inequality for plane closed curves, which states that if C is enclosed by a circle of radius r, then

$$L(C) \leqq r\kappa(C) . \tag{2}$$

The latter inequality gives a sharp result, with equality obtained in case C is a circle of radius r.

In this paper we sharpen (1) to the following result. Let C be a rectifiable closed curve enclosed by a $k-1$ dimensional sphere S of radius r in Euclidean k-space, $k \geqq 2$. Then

$$L(C) \leqq r\kappa(C) . \tag{3}$$

The proof of (3) again depends on the plane case and is motivated by the following construction. We form the cone T over the curve C with apex at the center of S, slit along a longest generator and develop the result in a plane. The resulting plane arc C' is completed to a closed plane curve C'' by attaching an arc of a circle. It is noted that the curvature of C' is equal pointwise to the geodesic curvature of C with respect to T, which in turn is not greater, pointwise, than the curvature of C. The length of C' is the same as that of C. The inequality (2) applied to C'' now gives (3).

2. In this section we prove some lemmas which lead directly to the main theorem.

LEMMA 1. *Let C be a rectifiable plane arc of length L. For any line G, let $n(p, \theta)$ be the number of intersections of G with C, where (p, θ), $p \geqq 0$, $0 \leqq \theta < 2\pi$, are the normal coordinates of G. Then*

$$L = \tfrac{1}{2}\int_0^{2\pi}\int_0^{\infty} n(p, \theta)\ dpd\theta . \tag{4}$$

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