HAUSDORFF DIMENSION OF LEVEL SETS OF SOME RADEMACHER SERIES

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1. Introduction. A special case of a result of Kaczmarz and Steinhaus [4] (Theorem 2 with $a = b$) shows that if $\{a_i\}$ $(i = 1, 2, \cdots)$ is a sequence of real numbers with $\sum_{i=1}^{\infty} |a_i| = +\infty$ and $a_i \to 0$, then the Rademacher series $\sum_{i=1}^{\infty} a_i R_i(x)$ assumes every preassigned real value c (cardinal number of the continuum) times for x in $(0, 1]$. One object of this paper is to refine this result in certain directions. We shall prove

THEOREM 1. *If the sequence $\{a_i\}$ is in l_2, but not in l_1, then $\sum_{i=1}^{\infty} a_i R_i(x)$ assumes every preassigned real value on a set of Hausdorff dimension* 1.

We shall also prove

THEOREM 2. *If $\{a_i\}$ is a sequence of bounded variation $(\sum_{i=1}^{\infty} |a_i - a_{i-1}| < \infty)$ which is not in l_1 but $a_i \to 0$, then $\sum_{i=1}^{\infty} a_i R_i(x)$ assumes each preassigned real value on a set of Hausdorff dimension at least* 1/2.

In § 6, we apply the method of proof to a problem on the distribution of digits in decimal expansions of numbers.

In § 7 through 11, we develop a theory of dimension of level sets for series of the type $\sum_{i=1}^{\infty} r^i R_i(x)$ where r is a fixed number in the interval $[1/2, 1)$.

2. Preliminary definitions and lemmas.

DEFINITION 1. The i^{th} $(i = 1, 2, \cdots)$ Rademacher function is defined to be $R_i(x) = 1 - 2\varepsilon_i(x)$ $(0 < x \leqq 1)$, where $\varepsilon_i(x)$ is the i^{th} digit of the (unique) nonterminating binary expansion of x.

DEFINITION 2. Let X be a subset of Euclidean n-space. Let $J_\varepsilon(X)$ be a finite or countably infinite set of open spheres $\{J_i\}$ $(i = 1, 2, \cdots)$ with finite diameters $|J_i|$ whose union covers X and whose diameters do not exceed ε where $\varepsilon > 0$. The Hausdorff outer measure of order s,

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