A SPECIAL CLASS OF MATRICES

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1. Introduction. Let D be an integral domain, K its quotient field, D^n the set of all n-by-1 matrices over D, and A an n-by-n matrix over a field containing K. We say that A has *property* P_D if and only if, for all nonzero u in D^n, the vector Au has at least one component in $D^* = D - \{0\}$. The setting in which this property arose is detailed in [**1**], where we investigated the case where D was either Z, the rational integers, or the ring of integers of an algebraic number field of class-number one. Now, if P is a permutation matrix, T is lower triangular with only ones in the diagonal, and N is nonsingular and over D, then $A = PTN$ has property P_D. It was shown in [**1**] that for $D = Z$ there are matrices not of the form PTN which have property P_D; but, at least in the case of the ring of integers of an algebraic number field of class-number one, we found the necessary but far from sufficient condition, that $\det A$ be in D^*. Our present purpose is to extend this to all algebraic number fields and also to prove necessary and sufficient conditions for property P_D in certain cases.

THEOREM I. *Let D be a domain whose quotient field K is algebraic over its prime field. Let A be an n-by-n matrix, where $n \leqq \sharp(K)$.*[1] *Then:*

(i) *If K is of prime characteristic, then A has property P_D if and only if $A = PTN$, where P, T and N are as above:*

(ii) *If D is Dedekind and K is a finite algebraic extension of the rationals, then for A to have P_D we must have* $\det A \in D^*$.

THEOREM II. *If $D = D_1[t]$, where t is transcendental over D_1, if $\sharp(D_1) > n$, and if A has P_D, then the rows of A can be so ordered that the matrices A_r of the first r rows of A have all r-by-r minors in D and not all zero, for $r = 1, 2, \cdots, n$. In particular, the first row is over D, and* $\det A \in D^*$.

If in addition we have only principal ideals, then we can reduce all but one element of the first row to zero and prove by induction:

COROLLARY. *If $D = F[t]$, where $\sharp(F) > n$, so K is a simple transcendental extension, then A has P_D if and only if $A = PTN$, where P, T and N are as above.*

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[1] $\sharp(K)$ = cardinality of K.