A GENERAL SOLUTION OF TONELLI'S PROBLEM OF THE CALCULUS OF VARIATIONS

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Summary. The *alpha-asymptotical* property is defined for integrals depending upon any number m of surfaces of any dimension in nonparametric form. The existence of an absolute minimum of any alpha-asymptotical and lower semicontinuous integral in any *regular* class of *varieties* which is closed with respect to the *m-uniform* (Tchebyshev) metric is proved.

1. DEFINITIONS. Let D_i $(i = 1, 2, \cdots, m)$ be a closed bounded set of the n-dimensional Euclidean space of the variable vector $x_i \equiv x_i^j$ $(j = 1, 2, \cdots, n)$, bounded by surfaces which are absolutely continuous in the sense of Tonelli [14, 16, 17], without multiple points, and let D be the cartesian product $\prod_{1i}^{m} D_i$. Let $y \equiv y_i$ $(i = 1, 2, \cdots, m)$ denote a vertical m-vector, and let p denote an mxn matrix, whose row-vectors are $p_i \equiv p_i^j$ $(j = 1, 2, \cdots, n)$. Let x be the mxn matrix whose row-vectors are x_i and $\phi[x, y, p]$ a real-valued function, defined for $x_i \in D_i$ $(i = 1, 2, \cdots, m)$ and for any y and p, which is continuous with all its derivatives of the types

$$\frac{\partial\phi[x, y, p]}{\partial p_r^s} \text{ and } \frac{\partial^2\phi[x, y, p]}{\partial p_r^s \partial p_r^t} \qquad (r = 1, \cdots, m; \quad s, t = 1 \cdots n) .$$

Let $q \leqq m$ be a positive integer and let U_q denote a set of q distinct positive integers out of the first m; let ζ be an index ranging over U_q, and let $\mu(\zeta)$ be a mapping of U_q into the set of the first n integers. It will be assumed throughout, that for every q, every U_q and every $\mu(\zeta)$, all the partial derivatives

$$\frac{\partial^{2q}\phi[x, y, p]}{\prod_{\zeta=1}^{q} \partial x_\zeta^{\mu(\zeta)} \partial p_\zeta^{\mu(\zeta)}} \tag{1.1}$$

exist and are continuous for every $x \in D$ and for every y and p.

Let T be a real positive number. Let $y(x) \equiv y_i(x_i)$ $(i = 1, 2, \cdots, m)$ denote a vector-valued function of the matrix x, such that each $y_i(x_i)$ is a function with values in $[-T, T]$, which only depends upon the row

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