COMPLETION OF MATHEMATICAL SYSTEMS

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1. Introduction. The completion problem to be considered may be informally and tentatively described as follows.

Let $\mathscr{A}$ be a class of systems of some type (e.g., $\mathscr{A}$ will be the class of all fields in Example 1; cf. §§ 6,7). For all $a, b \in \mathscr{A}$ let "$a \prec b$" mean that a is a subsystem (e.g., subfield in Example 1) of b. For each $a \in \mathscr{A}$ let $\pi(a)$ be a set of propositional forms[1] involving unknowns (e.g., polynomial equations in one unknown in Example 1); each of these forms may become a true or false proposition upon substitution of elements of a for the unknowns; a substitution turning a form into a true proposition is a solution of the form. For each $a \in \mathscr{A}$ let $\pi'(a)$ be the set of all members of $\pi(a)$ with solutions (relative to a). If $a, b \in \mathscr{A}$ and $a \prec b$, then each $p \in \pi(a)$ will correspond to some member, say $\rho_a^b(p)$, of $\pi(b)$ (e.g., if $\mathscr{A}$ is the class of all groups, the propositional form "$y^{-1}xy \neq x$ for some y in a" in unknown x could correspond to "$y^{-1}xy \neq x$ for some y in b"). We may say that $a \in \mathscr{A}$ is *complete* if and only if for each $b \in \mathscr{A}$ with $a \prec b$ and each $p \in \pi(a)$: if p has no solution (relative to a), then $\rho_a^b(p)$ has no solution (relative to b). (E.g., in Example 1, a field is complete if and only if it is algebraically closed.) The completion problem to be considered is: Does each $a \in \mathscr{A}$ have a complete extension?[2]

This extension problem will be formulated rigorously in §§ 5,6. In some explicit special cases in modern algebra the existence of a complete extension rests on (transfinitely) recursive definitions the justification of which at first glance would seem to require a very strong version of the axiom of choice (cf. Remark 5 of § 7). In this paper the set-theoretic foundations of such procedures will be examined. The result is a theorem from which will follow the usual extension theorems via the usual weak version of the axiom of choice.

2. Set-theoretic preliminaries. In axiomatic set theory one may consider the following versions of the axiom of choice.

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[1] Relative to (axiomatic) set theory, since a propositional form exists only in the metatheory, the propositional forms as such will have to be replaced by set-theoretic antecedents.

[2] This completion problem is a straightforward generalization of problems raised and solved by W. R. Scott [6] for groups (however, cf. Remark 5 of §7). The general problem of this paper will be illustrated by Scott's result via Example 2 (cf. §§6, 7).