THE SPECTRUM AND THE RADICAL IN LOCALLY m-CONVEX ALGEBRAS

ELEANOR KILLAM

Introduction. Let A be a complex algebra. A subset V of A is said to be *idempotent* if $VV \subseteqq V$; V is said to *m-convex* (multiplicatively-convex) if V is convex and idempotent; V is said to be *circled* if $cV = V$ for all scalars c such that $|c| \leqq 1$. A is a *locally convex algebra* if A is a (Hausdorff) topological algebra which has a basis for neighborhoods of the origin consisting of sets which are convex and circled; A is a *locally m-convex* (multiplicatively-convex) *algebra* if A has a basis for neighborhoods of the origin consisting of sets which are m-convex and circled. These sets can be taken to be closed. A family of closed, circled, m-convex subsets of a locally m-convex algebra A whose scalar multiples form a basis for the neighborhoods of the origin will be called an *m-base* for A.

If A is a locally m-convex algebra and $\mathscr{V}$ is an m-base for A define a nonnegative real-valued function $V(x)$ on A, V in $\mathscr{V}$, by $V(x) = \inf\{c > 0 : x \text{ in } cV\}$. Then $V(x+y) \leqq V(x) + V(y)$, $V(xy) \leqq V(x)V(y)$, and $V(cx) = |c|\, V(x)$ for all x, y in A and all scalars c. Thus with each V in $\mathscr{V}$ is associated a pseudo-norm $V(x)$. Denote the null set of $V(x)$ by N_V. Then N_V is a closed ideal and A/N_V is a normed algebra with norm $\bar{V}(x + N_V) = V(x)$. Denote A/N_V by A_V, $x + N_V$ by x_V, and the completion of A_V by B_V. Note that B_V is a Banach algebra.

1. The spectrum. An elements x in an algebra A is said to be *quasi-regular* in A if there exists an element y in A such that $x + y - xy = 0 = x + y - yx$. y is called the quasi-inverse of x. The *spectrum*, $Sp_A(x)$, of an element x in A, is given by $Sp_A(x) = \{c \neq 0\colon c \text{ complex}, c^{-1}x \text{ is not quasi-regular in } A\}$, with zero added unless A has an identity and x^{-1} exists in A. The *spectral radius*, $r_A(x)$, is defined by $r_A(x) = \sup\{|c| : c \text{ in } Sp_A(x)\}$. If A is a locally m-convex algebra and x is in A then $Sp_A(x)$ is not empty [1; 2.9].

Waelbroeck [3] has given a different definition of the spectrum of an element in a locally convex algebra with identity. Although his definition is actually given for a particular class of locally convex algebras, it can be used in any locally convex algebra with identity. However some of the properties which hold in the class of algebras which Waelbroeck considers may fail to hold in more general cases. We extend this definition to locally m-convex algebras which do not necessarily have

Received May 17, 1961. These results were part of the author's dissertation presented for the degree of Doctor of Philosophy in Yale University.