DIRECT DECOMPOSITIONS WITH FINITE DIMENSIONAL FACTORS

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The principal results. A fundamental theorem of Ore [10] states that if an element in a finite dimensional modular lattice is represented in two ways as a direct join of indecomposable elements, then the factors of the two decompositions are projective in pairs. The Krull-Schmidt theorem is an immediate consequence of this result. Subsequently many authors have considered direct decompositions in modular lattices. In particular, Kurosh [8, 9] and Baer [1, 2] obtained conditions which imply the existence of projective refinements of two direct decompositions of an element in an upper continuous modular lattice. When applied to the decompositions of a group G, the conditions of Kurosh and Baer are reflected in certain chain conditions on the center of G. In a somewhat different direction, Zassenhaus [11] has shown that the representation of an operator group as a direct product of arbitrarily many indecomposable groups each with a principal series is unique up to isomorphism.

This paper studies the direct decompositions of an element in an upper continuous modular lattice under the assumption that the element has at least one decomposition with finite dimensional factors. It is then shown that every other decomposition of the element refines to one with finite dimensional factors, and that a strong exchange isomorphism exists between two decompositions with indecomposable factors. This latter result sharpens the uniqueness result of Zassenhaus.

Before explicitly stating the principal results, let us note the following definitions. A lattice L is *upper continuous* if L is complete and

$$a \cap \bigcup_{k \in K} x_k = \bigcup_{k \in K} a \cap x_k$$

for every element $a \in L$ and every *chain* of elements x_k $(k \in K)$ in L.

If a and a_i $(i \in I)$ are elements of a complete lattice L with a null element 0, then a is said to be a *direct join* of the elements a_i $(i \in I)$, in symbols

$$a = \dot{\bigcup_{i \in I}} a_i ,$$

if $a = \bigcup_{i \in I} a_i$, and for each index $h \in I$ we have $a_h \cap \bigcup_{i \neq h} a_i = 0$. The direct join of finitely many elements $a_1, \cdots, a_n$ is also denoted by $a_1 \dot{\cup} \cdots \dot{\cup} a_n$. An element b is called a *direct factor* of a if $a = b \dot{\cup} x$ for some element x. An element a is *indecomposable* if $a \neq 0$ and $a =$

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