REMARKS ON AFFINE SEMIGROUPS

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A problem of fundamental importance in the study of measure semigroups is the following: if S is a compact topological semigroup and $\tilde{S}$ is the convolution semigroup (with the weak-$*$ topology) of nonnegative normalized regular Borel measures on S, what relationship exists between a measure μ in $\tilde{S}$ and its carrier? In the paper numbered [9, Lemma 5], Wendel proved that when S is a group and μ an idempotent in $\tilde{S}$, then carrier μ is a group and μ is Haar measure on carrier μ. He proved further that the mapping $\mu \to$ carrier μ is a one-to-one mapping from the set of idempotents of $\tilde{S}$ onto the set of closed subgroups of S. Glicksberg in [6] extended these results to the case when S is an abelian semigroup, In addition he showed (when S is a group or an abelian semigroup) the structure of the closed subgroups of $\tilde{S}$ to be quite simple: each closed subgroup of $\tilde{S}$ consists of the G-translates of Haar measure on some closed normal subgroup of a suitably chosen closed group G of S.

It is our purpose in this paper to prove in § 2 that these properties are equivalent in general, each being equivalent to several other properties of some interest (see Theorem 2). One of these conditions is the geometric requirement that $\tilde{S}$ can contain no 'parallelogram' whose vertices are μ, ν, $\mu\nu$, and $\nu\mu$, with all four of these measures idempotent and μ and ν distinct. A crucial lemma of independent interest is that found in Theorem 1 of § 1, where it is shown that a line segment of an affine semigroup (see [3] for definitions) which contains three distinct idempotents consists entirely of idempotents. Several corollaries are drawn from this theorem, among them the result that a compact affine semigroup consists of idempotents (i.e., is a *band* in the sense of [2]) if and only if it is rectangular, and that this occurs if and only if it is *simple* (i.e., contains no proper ideals).

References for terminology and notation used here may be found in [3, 6, 8, 9].

1. General affine semigroups. This section is devoted primarily to several results about general affine semigroups. However, we list first without proof two lemmas ([3, Theorem 3] and [4, Theorem 2]) needed in the sequel.

LEMMA 1. *Let T be a compact affine topological semigroup and*

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