ON HILBERT SPACE OPERATORS AND OPERATOR ROOTS OF POLYNOMIALS

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1. Introduction. The Cayley-Hamilton theorem states that any linear transformation (sometimes called "operator") on a finite dimensional vector space over a field is a root of its characteristic polynomial. On the other hand an operator on an infinite dimensional vector space need not be the root of any nonzero polynomial with scalar coefficients. It is our purpose to give necessary and sufficient conditions for a bounded operator on a complex Hilbert space to be the root of a nonzero polynomial with complex coefficients.

Significant in much linear algebra is the fact that an operator A on a finite dimensional vector space V over an algebraically closed field F must have an eigen value; more precisely, there is a scalar λ in F and a nonzero vector z in V such that $(A - \lambda)z = 0$. We make the following

DEFINITION. *An element λ in a field F is said to be an eigen value for the operator A on a (possibly infinite dimensional) vector space V over F if there exists at least one nonzero vector z in V such that $(A - \lambda)z = 0$. An operator A on V is said to be an eigen value producing (henceforth abbreviated "evp") operator if for each linear maniforld V' reducing A and $\neq V$, the operator A' induced by A on the quotient space V/V' has at least one eigen value.*

In particular if A is evp it has an eigen value, because (0) reduces A. One example of an evp operator is any operator on a finite dimensional vector space over an algebraically closed field. The central result of the present paper is that a bounded operator on a complex Hilbert space is evp if and only if it is the root of a nonzero polynomial with complex coefficients. Before we introduce Hilbert space operators we establish some algebraic machinery.

2. The structure of evp operators. In this section let A be an operator on a vector space V over the field F, and let V_λ be the linear manifold consisting of all vectors annihilated by some power of the operator $A - \lambda$, for each λ in F.

LEMMA 1. *Let λ and μ be distinct scalars and let z be a vector such that*

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