ON PERMUTATIONS INDUCED BY LINEAR VALUE FUNCTIONS

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1. Background and statement of the problem. Consider a set of n objects, symbolized by the integers

$$N = (1, \cdots, n) . \tag{1.1}$$

Let v_i be a real number to be called the *value* of object i ($i = 1, \cdots, n$). The *value of* $J = (i_1, \cdots, i_j) \subset N$ will mean

$$v(J) = \sum_{h=1}^{j} v_{i_h} , \tag{1.2}$$

and the *value* of the null set J_0 is

$$v(J_0) = 0 \tag{1.3}$$

The present paper is partly motivated by its bearing on linear programming problems in which a subset of N is sought, having a maximum value among all subsets satisfying some given restriction; for example, a condition of the form $\sum_{h=1}^{j} w_{i_h} < W$. In this restriction, w might be the weight of object i, and one would be seeking a subset of maximum value among those with a given upper bound on their total weights. In many applications, v_i and w_i are positive, but we do not impose this condition at present.

Let $\{J\}$ be the set of all the 2^n subsets of N. Given $J \in \{J\}$, we will denote with $[J]$ the set of all subsets of N each having the same value as J. Thus $\{J\}$ is partitioned into equivalence classes, each of the form

$$[J] = \{K \subset N \mid v(K) = v(J)\} . \tag{1.4}$$

These equivalence classes are ordered by the relation $\prec$, to be read *precedes*, defined thus:

$$[J] \prec [K] \quad \text{if} \quad v(J) < v(K) . \tag{1.5}$$

(A) We will denote by $\Pi(V; [J])$ the permutation of the equivalence classes in which they are arranged in order of increasing values; that is, $[J]$ comes before $[K]$ if $[J] \prec [K]$.

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