ON THE APPROXIMATION OF FUNCTION SPACES IN THE CALCULUS OF VARIATIONS

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Introduction. A basic feature of most of the methods used for the numerical calculation of a variational problem is the reduction of the infinite dimensional problem to a finite dimensional problem by some kind of approximation. One of the most natural approximations is that of replacing a curve or a surface by a finite number of points lying on or near the curve or surface. The points are then connected by simple arcs or surfaces, and the resulting approximation will, if the number of points is sufficiently large, presumably be close to the original curve or surface. The difficulties inherent in this approach to surface problems are well illustrated in the works of Rado [8], [9] on surface area.

The replacement of a curve by an approximating polygon, however, does lead to a usable finite dimensional approximation scheme. Lewy [3] (Chapter IV) gives a proof of the existence of an absolute minimum to the positive regular nonparametric problem by using such an approximation scheme, and his proof could be used to design a numerical process for approximating this minimum.

The methods of algebraic topology which M. Morse ([4] to [7]) applied to the calculus of variations have led to a greater understanding of the relationships between all the extremals to a variational problem. The extremals are classified according to their index types, in analogy with quadratic forms of a finite number of variables. While the extremals with nonzero index are not minimizing, they are of importance in many physical applications.

In this paper we shall treat the problem of computing the nonminimizing extremals as well as those of minimizing type, using the theory developed by Morse, together with a general theory of approximation. In part 1, a brief restatement of some of the principal definitions and theorems of Morse [6] will be given, in the current language of algebraic topology. In part 2, a general theory of approximation to an abstract metric space will be developed, and the convergence of the approximations to the critical levels of the problem defined on this space will be demonstrated. Part 3 will show that the polygonal approximations to curves leads, in the parametric problem, to approximations satisfying the theory of part 2.

The structure of part 2 is given with sufficient abstraction so that

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