A REMARK ON THE NIJENHUIS TENSOR

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The vanishing of the Nijenhuis tensor of the almost complex structure is known to give the integrability of the almost complex structure [3, 7]. In order to generalize this fact, we consider a vector 1-form h on a manifold M[4], whose Jordan canonical form at all points on M is equal to a fixed matrix μ. Following the idea of E. Cartan, we say that such a vector 1-form is 0-deformable [2]. The frames z at x such that $z^{-1}h_x z = \mu$ define a subbundle of the frame bundle over M, as x runs through M, and the subbundle is called a G-structure defined by h [1]. We find that for a certain type of 0-deformable h, the vanishing of the Nijenhuis tensor of h is sufficient for the G-structure to be integrable (Theorem, §2). In §5 we give an example of a 0-deformable derogatory nilpotent vector 1-form, whose Nijenhuis tensor vanishes, but whose G-structure is not integrable.

1. Vector forms and distributions. As usual, we begin by stating, that all the objects we encounter in this paper are assumed to be C^∞.

Let M be a manifold, T_x the tangent space at point x of M, T the tangent bundle over M, $T^{(p)}$ the vector bundle of tangential covariant p-vectors of M. A vector p-form is a cross-section of $T \otimes T^{(p)}$. The collection of all vector p-forms over M is denoted by Ψ_p. We notice that a vector 1-form is nothing but a law that assigns a linear transformation to each tangent space T_x at point x of M.

We list some definitions and lemmas of the theory of vector forms [4], which we use in the sequel.

If $P \in \Psi_p$, $Q \in \Psi_q$, then $P \barwedge Q \in \Psi_{p+q-1}$ is defined by

$$(1)\qquad (p \barwedge Q)(u_1, \cdots, u_{p+q-1}) = \frac{1}{(p-1)!\, q!} \sum_\alpha |\alpha| P(Q(u_{\alpha_1}, \ldots, u_{\alpha_p}), u_{\alpha_{p+1}}, \ldots, u_{\alpha_{p+q-1}})$$

where α runs through all the permutations of $(1, 2, \cdots, p+q-1)$, and $|\alpha|$ denotes the signature of the permutation α.

If h is a vector 1-form and P is a vector p-form, we write hP instead of $h \barwedge p$. In particular if $p = h$, we write $h \barwedge h$ as h^2. In general, $\underbrace{h \barwedge h \cdots \barwedge h}_{k \text{ times}}$ is written as h^k, and this agrees with the usual notation,

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