RINGS IN WHICH SEMI-PRIMARY IDEALS ARE PRIMARY

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Every ring considered in this paper will be assumed to be commutative and to have a unit element. An ideal A of a ring R will be called semi-primary if its radical $\sqrt{A}$ is prime. That a semi-primary ideal need not be primary is shown by an example in [3; p. 154]. This paper is a study of rings R satisfying the following condition: (*) Every semi-primary ideal of R is primary. The ring Z of integers clearly satisfies (*). More generally, if A is a semi-primary ideal of a ring R such that $\sqrt{A}$ is a maximal ideal of R, then A is primary. [3; p. 153]. Hence, every ring having only maximal nonzero prime ideals satisfies (*).

An ideal A of a ring R is called P-primary if A is primary and $P = \sqrt{A}$. If ring R satisfies (*), then A is $\sqrt{A}$-primary if and only if $\sqrt{A}$ is prime. Some well-known properties of a ring R satisfying (*) are listed below.

Property 1. If R satisfies (*) and A is an ideal of R, then R/A satisfies (*). [3; p. 148].

Property 2. If R satisfies (*), if A and B are ideals of R such that $A \subseteqq B \subseteqq \sqrt{A}$, and if A is $\sqrt{A}$-primary then B is $\sqrt{A}$-primary. [3; p. 147].

THEOREM 1. *If ring R satisfies (*) and P, A, and Q are ideals of R such that P is prime, $P \subset A$, and Q is P-primary, then $QA = Q$.*

Proof. Since $\sqrt{QA} = P$, QA is P-primary. Thus $Q \cdot A \subseteqq QA$ and $A \nsubseteq P$ imply that $Q \subseteqq QA \subseteqq Q$. Hence $QA = Q$ as asserted.

THEOREM 2. *If P is a nonmaximal prime ideal in a ring R satisfying (*) and if Q is P-primary, then $Q = P$.*

Proof. We let P_1 be a proper maximal ideal properly containing P. If $p_1 \in P_1$ such that $p_1 \notin P$ and if $p \in P$, then $Q \subseteqq Q + (pp_1) \subseteqq P$. By property 2, $Q + (pp_1)$ is P-primary. Since $pp_1 \in Q + (pp_1)$ and $p_1 \notin P$, $p \in Q + (pp_1)$. Then for some $q \in Q$, $r \in R$, $p(1 - rp_1) = q$. Now $1 - rp_1 \notin P_1$ since $P_1 \subset R$ so that $1 - rp_1 \notin P$. Thus $p \in Q$ and $P \subseteqq Q \subseteqq P$. Hence $P = Q$ and our proof is complete.

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