LIMITS AND BOUNDS FOR DIVIDED DIFFERENCES ON A JORDAN CURVE IN THE COMPLEX DOMAIN

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1. Introduction. Let $S_{n+1} = \{z_1, z_2, \cdots, z_{n+1}\}$ be a set of $n+1$ complex numbers and let f be a function on a set containing S_{n+1} to the complex numbers. The divided difference $d_n = d_n(f \mid z_1, z_2, \cdots, z_{n+1})$ of order n formed for the function f in the points[1] S_{n+1} is defined in a recursive manner as follows:

$$
\begin{aligned}
d_1 &= d_1(f \mid z_1, z_2) = \frac{f(z_1) - f(z_2)}{z_1 - z_2} \\
d_2 &= d_2(f \mid z_1, z_2, z_3) = \frac{d_1(f \mid z_1, z_2) - d_1(f \mid z_3, z_2)}{z_1 - z_3} \\
&\vdots \\
d_n &= d_n(f \mid z_1, z_2, \cdots, z_{n+1}) \\
&= \frac{d_{n-1}(f \mid z_1, z_2, \cdots, z_n) - d_{n-1}(f \mid z_{n+1}, z_2, \cdots, z_n)}{z_1 - z_{n+1}} .
\end{aligned}
$$

The definition requires further discussion when the points in S_{n+1} are not all distinct. We shall suppose that they are distinct unless provision is explicitly made for coincidences.

It can be proved by induction [7, p. 15] that if

$$\omega_{n+1}(z) = (z - z_1)(z - z_2) \cdots (z - z_{n+1}) ,$$

then

$$d_n = \sum_{k=1}^{n+1} \frac{f(z_k)}{\omega'_{n+1}(z_k)} , \tag{1.1}$$

where the prime denotes differentiation of $\omega_{n+1}(z)$ with respect to z. This formula shows that d_n is a symmetric function of $z_1, z_2, \cdots, z_{n+1}$.

The divided differences of a function given on the real line play a prominent role in the mathematics of computation. Their counterparts in the complex plane have appeared in various classical studies of approximation by complex polynomials. The formal algebra of complex

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[1] We use the words "points" and "numbers" interchangeably in referring to the arguments in divided differences. This follows the practice in interpolation theory. It is consistent within this terminology to speak of "coincident points" z_k.