ON FUNDAMENTAL PROPERTIES OF A BANACH SPACE WITH A CONE

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1. Introduction. Normed vector lattices have been investigated from various angles (see [1] Chap. 15 and [7] Chap. 6). On the contrary, it seems that there remain several problems unsolved in the theory of general normed spaces with a cone since the pioneer works of Riesz and Krein, though recently Namioka [8], Schaefer [9] and others made many efforts in analysing and extending the results of Riesz and Krein. In this paper we shall discuss two among them. Let E be a Banach space with a closed cone K (for the terminologies see § 2);

(A) What condition on the dual E^* is necessary and sufficient for that $E = K - K$?

(B) What condition on the dual is necessary and sufficient for the interpolation property of E?

Grosberg and Krein [3] dealt with (A) in a reversed form;

(A′) What condition on E is necessary and sufficient for that $E^* = K^* - K^*$ where K^* is the dual cone?

Schaefer ([9], Th. 1.6) obtained a complete answer to (A′) within a scope of locally convex spaces. A result of Riesz gives a half of an answer to (B), while Kreín [6] obtained a complete answer only under the assumption that the cone has an inner point.

The purpose of this paper is to give answers to both (A) and (B) in natural settings. Our starting assumptions consist of the completeness of E and of the closedness of the cone K.

After several comments on order properties in § 2, Lemmas in § 3 present algebraic forms to both the property named normality by Krein [5] and that named (BZ)-property by Schaefer [9], supported by Banach's open mapping theorem. Then Theorem 1 will produce an answer to (A) via these Lemmas. § 4 is devoted to an answer to (B) under the condition that E is an ordered Banach space. It should be remarked that our main theorems are also valid for (F) spaces, that is, metrisable complete locally convex spaces.

2. Definitions and consequences. Let E[1] be a real normed space and let K be a cone, that is, a subset of E with the following properties:

(1) $K + K \subset K$,

(2) $\alpha K \subset K$ for all $\alpha \geqq 0$, and

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[1] Elements of E are denoted by $x, y, a, \cdots, e$, and those of the dual E^* by f, g, h. Scalars are denoted by Greek letters. θ is reserved for the zero element.