ON CONVEX HULLS OF TRANSLATES

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1. Let G be a locally compact group, g a closed subgroup, each taken with its left invariant Haar measure. For f in $L_1(G)$ let f^s denote a right translate of $f(f^s(x) = f(xs))$ and let C_f denote the closed convex hull of the set $\{f^s: s \in g\}$ of right translates of f by elements of g.

Recently Reiter [9] considered the problem of determining the distance in L_1 from the origin to C_f and proved, when g is abelian[1] and the homogeneous space G/g of left cosets xg has a left invariant measure, that the distance is given by the expression

$$\int_{G/g} \left| \int_g f(xs)ds \right| dx' , \tag{1.1}$$

where integration is with respect to the invariant measures (suitably normalized), and x is an element of the coset x'. Now this suggests the following (overly general) question: suppose one has a semigroup S of operators of norm 1 on a Banach space B; under what sort of conditions can one explicitly determine the distance from the origin to the convex hull of the orbit Sx of an x in B?

In the present note we give a simple approach to certain problems of this sort (Lemma 2.1), which yields some information whenever S, in the terminology of [3], is right amenable, and leads to an explicit determination of the distance in a variety of cases (see 2.2–2.4). In particular we obtain (in 3.2) a considerable strengthening of Reiter's result in which g assumes the rôle of a very well behaved transformation group on a locally compact space G, while Haar measure on G is replaced by an essentially g-invariant measure.

But for a recent extension by Day [4] of the Markov-Kakutani theorem [1, p. 114; 5, p. 456] we should have to take our semigroup S abelian; for the reader's convenience we shall begin by giving a (rather different) proof of Day's result.

2. Let S be a semigroup, $m(S)$ the usual supremum normed space of all bounded complex functions on S. For $f \in m(S)$ let $f_s(t) = f(st)$, $s, t \in S$. A *left invariant mean* M on $m(S)$ is a nonnegative (hence continuous) linear functional for which[2]

$$\langle 1, M \rangle = 1, \langle f_s, M \rangle = \langle f, M \rangle$$

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[1] Or somewhat more generally, the product of a compact and an abelian group.

[2] Right invariant means are defined analogously, with $\langle f^s, M \rangle = \langle f, M \rangle$.