SOME METRICS ON THE SUBSPACES OF A BANACH SPACE

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Notation, Terminology and Conventions. We shall denote by $[X]$ the Banach algebra of all bounded operators mapping a Banach space X into itself. Unless otherwise stated, all operators are defined everywhere and bounded, and all convergence of operators is with respect to the uniform operator topology. An invertible element of $[X]$ will be called an invertible operator or an automorphism. A linear homeomorphism of one Banach space onto another will be called an isomorphism. The symbol "I" will be used for the identity operator. The term "subspace" will mean "closed linear manifold," and given a subspace Y, we will denote by $\Sigma(Y)$ the set, $\{y \in Y \mid \|y\| = 1\}$. Finally, the set of subspaces of a Banach space X will be denoted by S_X. Additional terminology and notation will be developed as needed.

Introduction. This paper is devoted to the study of three metrics on the set of subspaces of a Banach space: one due to J. J. Schäffer (see §1), one obtained as a modification of the opening (see § §2 and 3), and one due to J. D. Newburgh (see §7). I am indebted to Dr. J. J. Schäffer for helpful conversations and suggestions, and for the elegance of the demonstration in §5. In addition, it was a desire on my part to find geometric properties of his metric which led me to compare it with the opening. It turns out that this metric has strong connections with the opening. The connections between the three metrics, as well as properties of the opening of interest in themselves, form the subject matter of this paper.

1. The metric of Schäffer. In this section we list some of Schäffer's results surrounding the definition of his metric. This metric is introduced and discussed in [11].

DEFINITION. Let X be a Banach space. For arbitrary subspaces Y, Z we define:

$$r_0(Y, Z) = \begin{cases} \inf \{\|C - I\| \mid C \text{ is an invertible operator and } CY = Z\}, \\ \qquad \text{if such an operator } C \text{ exists.} \\ 1, \text{ if no such operator } C \text{ exists.} \end{cases}$$

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