EXISTENCE AND ASYMPTOTIC BEHAVIOR OF PROPER SOLUTIONS OF A CLASS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. This paper deals with proper solutions of the second-order nonlinear differential equation

$$y'' = yF(y, x)\ , \tag{1.1}$$

where (i) $F(u, x)$ is continuous in u and x for $0 \leqq u < +\infty$ and $x \geqq x_0$,

(ii) $F(u, x) > 0$ for $u > 0$ and $x \geqq x_0$,

(iii) $F(u, x) < F(v, x)$ for each $x \geqq x_0$ and $0 < u < v < +\infty$.

By a proper solution we understand a real-valued solution y of (1.1) which is of class $C^2[a, \infty)$, where $x_0 \leqq a < +\infty$. An example of equations of this type is the Emden-Fower equation [2, chapter 7]

$$y'' = x^\lambda y^n\ . \tag{1.2}$$

Our interest is in the existence and asymptotic behavior of *positive proper solutions* of (1.1). Since $F(y, x) > 0$ for $y > 0$, all positive solutions of this equation are convex. They are therefore of two types: (1) those which are monotonically decreasing and tending to nonnegative limits as $x \to +\infty$, and (2) those which are ultimately increasing and becoming unbounded as x becomes infinite.

In this section we shall consider proper solutions which are of type (1), i.e., solutions which are confined to the semi-infinite strip $S = \{(x, y)\colon 0 \leqq y \leqq K, a \leqq x < +\infty\}$. We observe that in view of properties (i) and (iii) the function $yF(y, x)$ satisfies a Lipschitz condition

$$|\,uF(u, x) - vF(v, x)\,| \leqq H|\,u - v\,| \tag{1.3}$$

in every closed rectangle $R = \{(x, y)\colon 0 \leqq y \leqq K, a \leqq x \leqq b\}$, where $H = H(K, a, b)$. Before taking up the existence of such solutions, we first derive the following lemmas.

LEMMA 1.1. *Let $u(x)$ be a nonnegative solution of* (1.1) *passing*

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