ON AN INEQUALITY OF P. R. BEESACK

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In a recent paper [1], P. R. Beesack derived the inequality

$$|g(x, s)| \leqq \frac{\prod_{\nu=1}^{n} |x - a_\nu|}{(a_n - a_1)(n-1)!} \tag{1}$$

for the Green's function $g(x, s)$ of the differential system

$$\begin{gathered} y^{(n)} = 0 , \qquad y(a_\nu) = 0 , \qquad \nu = 1, 2, \cdots, n, \\ -\infty < a_1 < a_2 < \cdots < a_n < \infty . \end{gathered} \tag{2}$$

In addition to being interesting in its own right, this inequality is a useful tool in the study of the oscillatory behavior of nth order differential equations. It would therefore appear to be worth while to give a short proof of (1). The derivation of this inequality in [1] is rather complicated.

We denote by $[x_0, x_1, \cdots, x_k]$ the kth difference quotient of the function $g(x) = g(x, s)$, i.e., we set

$$[x_0, x_1] = \frac{g(x_0) - g(x_1)}{x_0 - x_1} ,$$

$$[x_0, x_1, \cdots, x_\nu] = \frac{[x_0, x_1, \cdots, x_{\nu-1}] - [x_1, x_2, \cdots, x_\nu]}{x_0 - x_\nu} , \qquad \nu = 2, \cdots .$$

This difference quotient can also be represented in the form

$$[x_0, \cdots, x_k] = \int \cdots \int g^{(k)}(t_0 x_0 + t_1 x_1 + \cdots + t_k x_k) dt_0 dt_1 \cdots dt_{k-1} , \tag{3}$$

where the integration is to be extended over all the positive values of the t_ν for which

$$t_0 + t_1 + \cdots + t_k = 1 . \tag{4}$$

This formula, which goes back to Hermite, is easily verified by induction (cf., e.g., [2]). It holds if $g(x)$ has continuous derivatives up to the order $k-1$, and if $g^{(k)}$ is piecewise continuous.

Since, by its definition, $g(x, s)$ has continuous derivatives up to the order $n-2$, while $g^{(n-1)}$ has the jump

$$g_+^{(n-1)}(s) - g_-^{(n-1)}(s) = -1 \tag{5}$$

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