STRONGLY RECURRENT TRANSFORMATIONS

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Let $(X, \mathscr{B}, m)$ be a finite or σ-finite and non-atomic measure space. A set B is said to be measurable if it is a member of $\mathscr{B}$. Two measures on $\mathscr{B}$, finite or σ-finite (one may be finite and the other σ-finite), are said to be equivalent if they have the same null sets. In this paper we consider a one-to-one, nonsingular, measurable transformation ϕ of X onto itself. By a nonsingular transformation ϕ we mean $m(\phi B) = m(\phi^{-1}B) = 0$ for every measurable set B with $m(B) = 0$, and by a measurable transformation ϕ we mean $\phi B \in \mathscr{B}$ and $\phi^{-1}B \in \mathscr{B}$ for every $B \in \mathscr{B}$. We shall say that the transformation ϕ is measure preserving (with respect to a measure μ) or equivalently, μ is an invariant measure (with respect to the transformation ϕ) if $\mu(\phi B) = \mu(\phi^{-1}B) = \mu(B)$ for every measurable set B.

A recurrent transformation is a common notion in ergodic theory. This is a measurable transformation ϕ defined on a finite or σ-finite measure space $(X, \mathscr{B}, m)$ with the following property: if A is a measurable set of positive measure, then for almost all $x \in A$ $\phi^n x$ belongs to A for infinitely many integers n. It is not difficult to see that every measurable transformation which preserves a finite invariant measure μ equivalent to m is recurrent. The converse statement is not in general true; for example an ergodic transformation which preserves an infinite and σ-finite measure is always recurrent yet it does not preserve a finite invariant equivalent measure. In this paper we restrict the notion of a recurrent transformation. We introduce the notion of a strongly recurrent set and define a strongly recurrent transformation. We show that a transformation ϕ is strongly recurrent if and only if there exists a finite invariant measure μ equivalent to m (Theorem 2). This is accomplished by showing the connection between strongly recurrent sets and weakly wandering sets (Theorem 1). Weakly wandering sets were introduced in [1], and the condition that a transformation ϕ does not have any weakly wandering set of positive measure was further strengthened (see condition $(W)^*$ below). It was shown in [1] that this stronger condition was again a necessary and sufficient condition for the existence of a finite invariant measure μ equivalent to m. We show that a similar strengthening for a strongly recurrent transformation is false for a wide class of measure preserving transformations defined on a finite measure space (Theorem 3).

DEFINITION. A measurable set S is said to be *strongly recurrent*

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