AN APPROXIMATE GAUSS MEAN VALUE THEOREM

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1. Introduction. The mean value theorem of Gauss, and its converse, due to Koebe, have long been known to characterize harmonic functions. Since any second order homogeneous elliptic operator L can, by an appropriate linear change of variables, be reduced (at a given point) to the Laplacian, it seems reasonable to expect that solutions of $Lu = 0$ should, when averaged over appropriate small ellipsoids, satisfy an approximate Gauss-type theorem, and one could hope that such a mean value property would characterize the solutions of the equation.

It turns out that this is the case. In fact the operator need not be elliptic, but may be parabolic, or of mixed elliptic and parabolic type. While the methods used here do not permit the weak smoothness conditions on the solutions admitted by Koebe's theorem, the result is stronger than might be expected in that no smoothness, not even measurability, is required of the coefficients of L: they need only be defined.

Since the result applies to parabolic equations, it seems of interest to examine the heat equation, for it can be cast in the required form. This leads to a characterization of its solutions in terms of averages over parabolic arcs.

2. The basic theorem. In the following $D_i = \partial/\partial y_i$, $D_{ij} = \partial^2/\partial y_i \partial y_j$, $u_{,ij} = D_{ij}u$, and ∇_y is the gradient operator with respect to the components of y.

It is convenient to consider equations of the form $Lu = f$, where f need only be defined, and may depend on u and any of its derivatives.

LEMMA. *Let $A = [a_{ij}]$ be an $n \times n$ constant nonnegative definite symmetric matrix, and denote by $B = [b_{ij}]$ the unique nonnegative definite symmetric square root of A. Let u be defined in a neighborhood of a point y in E_n, and be twice differentiable at y. For this y define the quadratic function q of x by*

$$q(x) \equiv (Bx \cdot \nabla_y)^2 u(y) .$$

Then the sum of the coefficients of the squared terms of $q(x)$ is $\sum_{i,j} a_{ij} u_{ij}(y)$.

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