A NOTE ON ORTHOGONAL LATIN SQUARES

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1. Introduction. The purpose of this note is to give an improved estimate for $N(n)$, the maximal number of pairwise orthogonal Latin squares, by following the method of Chowla, Erdös and Straus [2]. The difference is that we use a result of Buchstab [1] rather than that of Rademacher in the sieve argument. Our result is that if c is any number less than 1/42, then for all large n we have $N(n) > n^c$.

In the notation of Buchstab, write $P_\omega(x; x^{1/a})$ for the number of positive integers not exceeding x which do not lie in any of the progressions a_0 mod p_0, a_i mod p_i, or b_i mod p_i, where $p_0 = 2$, and p_i runs over the primes from 3 to $x^{1/a}$. The subscript ω refers to the fact that P depends on the a_i, b_i. Buchstab proves that

$$P_\omega(x; x^{1/a}) > \lambda(a) \frac{c'x}{(\log x)^2} + 0\left(\frac{x}{(\log x)^3}\right), \tag{1}$$

where c' is a constant 0.4161 and $\lambda(5) \geqq 0.96$.

The properties of $N(n)$ used for the proof are those of [2]:

A. $N(ab) \geqq \mathrm{Min}\,\{N(a), N(b)\}$.

B. $N(n) \leqq n - 1$, with equality when n is a prime-power.

C. If $k \leqq 1 + N(m)$ and $1 < u < m$, then

$$N(u + km) \geqq \mathrm{Min}\,\{N(k), N(k+1), 1 + N(m), 1 + N(u)\} - 1.$$

We note that A and B are due to H. F. MacNeish, while C was found by Bose and Shrikhande.

2. Lower estimation of $N(n)$. We must deal separately with odd n and even n, and we use a fact proven in [1], called there "Lemma D":

D. The number of integers no greater than x, which have a prime factor in common with n and greater than n^g, is no greater than x/gn^g.

Estimate for even n. We pick k so that

$$\begin{cases} k \equiv -1 \pmod{2^{[\log_2 n/\alpha]}}, \\ k \not\equiv 0 \text{ or } -1 \pmod{p} \text{ for } 3 \leqq p \leqq n^{1/\beta}, \\ k \leqq n^{1/\gamma}. \end{cases} \tag{2}$$

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