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APPROXIMATION BY CONVOLUTIONS

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This paper is concerned mainly with approximating functions on closed subsets P of a locally compact Abelian group G by absolute-convex combinations of convolutions $f * g$, with f and g extracted from bounded subsets of conjugate Lebesgue spaces $L^p(G)$ and $L^{p'}(G)$. It is shown that the Helson subsets of G can be characterised in terms of this approximation problem, and that the solubility of this problem for P is closely related to questions concerning certain multipliers of $L^p(G)$. The final theorem shows in particular that the P. J. Cohen factorisation theorem for $L^1(G)$ fails badly for $L^p(G)$ whenever G is infinite compact Abelian and $p > 1$.

1. The Approximation Problem.

(1.1) Throughout this note, G denotes a locally compact Abelian group and X its character group. For the most part we shall be concerned with the possibility of approximating functions on closed subsets P of G by absolute-convex combinations

$$\sum_{r=1}^{n} \alpha_r (f_r * g_r) \,, \tag{1}$$

of convolutions $f * g$, where f and g are selected freely from bounded subsets of conjugate Lebesgue spaces $L^p(G)$ and $L^{p'}(G)$ $(1/p + 1/p' = 1)$. In the sums (1), the number n of terms is variable, whilst the complex coefficients α_r are subject to the condition

$$\sum_{r=1}^{n} | \alpha_r | \leqq 1 \,. \tag{2}$$

Accordingly, if the f_r and g_r are respectively free to range over subsets A and B of $L^p(G)$ and $L^{p'}(G)$, the allowed sums (1) compose precisely the convex, balanced envelope of

$$A * B = \{ f * g : f \in A, g \in B \} \,.$$

We denote by $C_0(G)$ the Banach space of continuous, complex-valued functions on G which tend to zero at infinity, the norm being $\| u \| = \sup \{ | u(x) | : x \in G \}$. The space $C_0(P)$ is defined similarly, P replacing G throughout. If G (or P) is compact, the restriction that the functions tend to zero at infinity becomes void; we then write $C(G)$ (or $C(P)$) in place of $C_0(G)$ (or $C_0(P)$).

It is well-known that if $1 < p < \infty$ then $f * g \in C_0(G)$ whenever

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