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LAPLACE'S METHOD FOR TWO PARAMETERS

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The behavior for large h and k of the integral

$$I(h, k) = \int_0^a f(t) \exp [- h\phi(t) + k\psi(t)]\, dt$$

is considered under hypotheses which are fulfilled, for example, if f, ϕ, ψ are real analytic, ϕ is strictly increasing, and $\phi(0) = \psi(0) = 0$. In most cases it is assumed that $k = o(h)$ as $h, k \to \infty$. If ν and μ are the respective orders of the first nonvanishing derivatives of ϕ and ψ at the origin, it is found that the behavior of $I(h, k)$ depends on whether:

(1) $0 < \liminf k^\nu h^{-\mu}$ and $\limsup k^\nu h^{-\mu} < \infty$,

(2) $k^\nu h^{-\mu} \to 0$, (3) $k^\nu h^{-\mu} \to \infty$ and $\psi^{(\mu)}(0) < 0$, or

(4) $k^\nu h^{-\mu} \to \infty$ and $\psi^{(\mu)}(0) > 0$.

In case (1) it is shown that $I(h, k)$ is asymptotic to a power series in $(k/h)^{1/(\nu-\mu)}$ with coefficients depending on $k^\nu h^{-\mu}$. In case (2) it is shown that $I(h, k)$ is asymptotic to a double power series in $h^{-1/\nu}$ and $kh^{-\mu/\nu}$. In case (3) it is shown that $I(h, k)$ is asymptotic to a double power series in $k^{-1/\mu}$ and $hk^{\nu-\mu}$. In case (4) it is shown that there exist two parameters σ, τ tending to zero as $h, k \to \infty$ such that $\exp(\sigma^{-2})\, I(h, k)$ is asymptotic to a double power series in σ and τ. If $\mu \leqq \nu$ it is proved that the coefficients of the above power series are unique.

It is the purpose of this paper to obtain asymptotic expansions of the integral $I(h, k)$, for $a > 0$, as $k, h \to \infty$. In most cases we assume that h and k are bound by the relation $k = o(h)$. We assume, roughly speaking, that $\phi(t) \sim a_0 t^\nu$ $(a_0 > 0)$, $\psi(t) \sim b_0 t^\mu$, and $f(t) \sim c_0 t^\lambda$ as $t \to 0$. If $k = 0$ and $\nu = 2$ this is the classical Laplace's Method. We will show that the problem divides naturally into four cases: $k^\nu h^{-\mu} \to 0$, $k^\nu h^{-\mu} \to \infty$ $(b_0 < 0)$, $k^\nu h^{-\mu} \to \infty$ $(b_0 > 0)$, and $k^\nu h^{-\mu}$ is bounded away from both zero and infinity. Tricomi [4] and Fulks [3] have obtained results along this line when $\nu = 2$, $\mu = 1$, and $\lambda = 0$. Tricomi considered a specific integral of this type (related by a change of variable to the incomplete gamma function) and obtained complete expansions in three of the four above cases. Fulks considered a general class of integrals and obtained the first term in all four cases. The methods of both authors depend quite strongly on the quadratic nature of the exponent near the origin. In this paper we will consider aribtrary ν, μ, λ and obtain complete asymptotic expansions in all four cases. The

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