PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

ISOMORPHIC GROUPS AND GROUP RINGS

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Let $\mathfrak{G}$ be a finite group, S a commutative ring with one and $S[\mathfrak{G}]$ the group ring of $\mathfrak{G}$ over S. If $\mathfrak{H}$ is a group with $\mathfrak{G} \cong \mathfrak{H}$ then clearly $S[\mathfrak{G}] \cong S[\mathfrak{H}]$ where the latter is an S-isomorphism. We study here the converse question: For which groups $\mathfrak{G}$ and rings S does $S[\mathfrak{G}] \cong S[\mathfrak{H}]$ imply that $\mathfrak{G}$ is isomorphic to $\mathfrak{H}$?

We consider first the case where $S = K$ is a field. It is known that if $\mathfrak{G}$ is abelian then $Q[\mathfrak{G}] \cong Q[\mathfrak{H}]$ implies that $\mathfrak{G} \cong \mathfrak{H}$ where Q is the field of rational numbers. We show here that this result does not extend to all groups $\mathfrak{G}$. In fact by a simple counting argument we exhibit a large set of nonisomorphic p-groups with isomorphic group algebras over all noncharacteristic p fields. Thus for groups in general the only fields if interest are those whose characteristic divides the order of the group.

We now let $S = R$ be the ring of integers in some finite algebraic extension of the rationals. We show here that the group ring $R[\mathfrak{G}]$ determines the set of normal subgroups of $\mathfrak{G}$ along with many of the natural operations defined on this set. For example, under the assumption that $\mathfrak{G}$ is nilpotent, we show that given normal subgroups $\mathfrak{M}$ and $\mathfrak{N}$, the group ring determines the commutator subgroup $(\mathfrak{M}, \mathfrak{N})$. Finally we consider several special cases. In particular we show that if $\mathfrak{G}$ is nilpotent of class 2 then $R[\mathfrak{G}] \cong R[\mathfrak{H}]$ implies $\mathfrak{G} \cong \mathfrak{H}$.

1. Remarks on group algebras.

Recently examples have been given of pairs of groups $\{\mathfrak{G}, \mathfrak{H}\}$ for which $K[\mathfrak{G}]$ is K-isomorphic to $K[\mathfrak{H}]$ for all fields K whose characteristic does not divide the order of the groups. We show here by a simple counting argument that this is not particularly surprising. This approach was suggested by Professor R. Brauer.

We prove

THEOREM A. *Suppose $Q[\mathfrak{G}] \simeq Q[\mathfrak{H}]$ where Q is the field of rational numbers. Then for all fields K whose characteristic does not divide $|\mathfrak{G}| = |\mathfrak{H}|$, the order of the groups, we have $K[\mathfrak{G}[\simeq K[\mathfrak{H}]$.*

THEOREM B. *There exists a set of $p^{B(n)}$ nonisomorphic groups of order p^n where $B(n) = 2/27\,(n^3 - 17\,n^2)$ which have isomorphic group algebras over all noncharacteristic p fields.*

Received January 28, 1964.