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EXTREME OPERATORS INTO $C(K)$

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If X and Y are real Banach spaces let $S(X, Y)$ denote the convex set of all linear operators from X into Y having norm less than or equal to 1. The main theorem is this: If K_1 and K_2 are compact Hausdorff spaces with K_1 metrizable and if T is an extreme point of $S(C(K_1), C(K_2))$, then there are continuous functions $\phi: K_2 \to K_1$ and λ in $C(K_2)$ with $|\lambda| = 1$ such that $(Tf)(k) = \lambda(k)f(\phi(k))$ for all k in K_2 and f in $C(K_1)$. There are several additional theorems which discuss the possibility of replacing $C(K_1)$ in this theorem by an arbitrary Banach space.

Suppose that K_1 is a compact Hausdorff space and that $C(K_1)$ is the Banach space of real-valued continuous functions on K_1, with supremum norm. Denote by S^* the unit ball of $C(K_1)^*$; then S^* is a weak* compact convex set and therefore the set $\operatorname{ext} S^*$ of its extreme points is nonempty, by the Krein-Milman theorem. Arens and Kelley [1] (cf. [4, p. 441]) showed that these extreme points are precisely those functionals of the form $f \to \lambda f(k)(f \in C(K_1))$, where $k \in K_1$ and $\lambda = 1$ or $\lambda = -1$. [We denote the functional "evaluation at k" by φ_k, so the extreme points of S^* are the functionals $\lambda\varphi_k$, $|\lambda| = 1$.] If we restrict our attention to the "positive face" of S^* (those functionals φ such that $\varphi(1) = 1$), then this is a weak* compact convex subset of S^* and its extreme points are those obtained from $\operatorname{ext} S^*$ by taking $\lambda = 1$. Now, the members of $C(K_1)^*$ can be regarded as continuous linear operators from $C(K_1)$ into $C(K_2)$, where K_2 consists of a single point. Thus, it is natural to consider the possible extension of the above results to the more general situation when K_2 is an arbitrary compact Hausdorff space, and S^* is replaced by the unit ball $S = S(C(K_1), C(K_2))$ of the Banach space $B = B(C(K_1), C(K_2))$ of all bounded linear operators from $C(K_1)$ into $C(K_2)$, with the usual operator norm. Corresponding to the positive face of S^* we have the convex subset S_1 of S, consisting of those T in S such that $T1 = 1$. It was shown by A. and C. Ionescu Tulcea [5] (and generalized in [8]) that $\operatorname{ext} S_1$ consists of the "composition operators", i.e., those of the form $(Tf)(k) = f(\psi(k))$ $(f \in C(K_1), k \in K_2)$ where ψ is a continuous function from K_2 into K_1. It is easily seen that these operators are also extreme in S; more generally, if $\psi: K_2 \to K_1$ is continuous and $\lambda \in C(K_2)$ with $|\lambda| = 1$,

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