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SOME ASPECTS OF TORSION

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Using S. E. Dickson's characterization of a torsion class, a class of modules closed under taking factors extensions and arbitrary direct sum, we study torsion classes closed under taking submodules and arbitrary direct products. We show that these classes are in one-to-one correspondence with idempotent two sided ideals of the ring. Finally we investigate the structure of rings R for which the torsion class $\mathscr{T}_0 = \{M \mid \mathrm{Hom}_R(M,Q(R)) = 0,\ Q(R)$ the minimal injective for $R\}$ is closed under taking products.

The original purpose of this paper was to show that for certain rings the direct product of torsion modules is again torsion, and by torsion we mean a particular kind of torsion defined in § 3. However, we find that S. E. Dickson [3] has given a set of axioms for torsion theories in Abelian categories and we thought it best to work within his axiomatic system.

In § 1 we summarize the work of Dickson and study, within the context of modules over a ring, torsion theories closed under taking submodules. In § 2, we give a complete characterization of all torsion theories closed under taking submodules and direct products. Finally, in § 3 we show that a fairly wide class of rings enjoy the property that a particular kind of torsion is closed under taking submodules and direct products.

1. Sets of torsion theories.

Dickson [3] has introduced a set of axioms for torsion theories in certain abelian categories sufficiently general to include the category ${}_R\mathscr{M}$ of left modules and homomorphisms over a ring R with identity. His axioms (stated here for ${}_R\mathscr{M}$) are as follows.

A torsion theory for ${}_R\mathscr{M}$ is a pair $(\mathscr{T}, \mathscr{F})$ of classes of modules such that:

I. $\mathscr{T}$ and $\mathscr{F}$ have only 0 in common.

II. $\mathscr{T}$ is closed under taking factors and $\mathscr{F}$ is closed under taking submodules.

III. For each M in ${}_R\mathscr{M}$ there exists a unique submodule $T(M)$ of M such that

$$0 \to T(M) \to M \to M/T(M) \to 0$$

is exact with $T(M) \in \mathscr{T}$ and $M/T(M) \in \mathscr{F}$.

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