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SINGULARITIES IN A VARIATIONAL PROBLEM WITH AN INEQUALITY

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The subject of the paper is the variational problem of Lagrange with an inequality in the form (a) $\phi(x, y) \geqq 0$ or (b) $\phi(x, y, y') \geqq 0$. The question of existence and uniqueness of the continuation of a minimizing arc is investigated at points of the boundary $\phi = 0$. Various phenomena, including splitting of extremals, dead-end, entry into, and exit from the boundary, are treated and the conditions for their occurrence are derived. The nature of the continuation is related to the "index" associated with an extremal.

An appendix extends the results to a control problem of the Mayer type.

In the variational problem of Lagrange with the inequality (a) $\phi(x, y) \geqq 0$ or (b) $\phi(x, y, y') \geqq 0$, case (a) has been treated by Bolza [3] and Mancil [6], and case (b) by Valentine [8]. Despite the relative antiquity of the problem several questions have remained unresolved.

A difficulty arises when an extremal of the problem has no unique continuation. We distinguish continuations in the region $\phi > 0$, and continuations in the boundary $\phi = 0$. Let the type of continuation not be specified *a priori*, and let H denote the corresponding Hilbert determinant of the composite arc. As will be shown, in case (a)

$$H = \begin{vmatrix} f_{y'y'} & -\phi_y \\ 0 & \phi \end{vmatrix}$$

and $H = 0$ if and only if $\phi = 0$; in case (b)

$$H = \begin{vmatrix} F_{y'y'} & \phi_{y'} \\ \lambda\phi_{y'} & \phi \end{vmatrix},$$

where λ is a Lagrange multiplier and $F \equiv f + \lambda\phi$, and $H = 0$ if and only if $\phi = \lambda = 0$. Since a solution generally contains points of the boundary, clearly the singularities defined by $H = 0$ deserve attention. For not only do they arise in practical problems, as noted by Garfinkel [5] and others, but they are also of intrinsic mathematical interest.

A systematic treatment of such singularities is undertaken here. Various phenomena, including splitting of extremals, dead-end, entry into, and exit from the boundary, will be treated and the conditions for their occurrence derived. It will be shown that case (a) exhibits the splitting and the dead-end, in contrast case (b), where a unique

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