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NILPOTENCE OF THE COMMUTATOR SUBGROUP IN GROUPS ADMITTING FIXED POINT FREE OPERATOR GROUPS

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Let V be a group of operators acting in fixed point free manner on a group G and suppose V has order relatively prime to $|G|$. Work of several authors has shown that if V is cyclic of prime order or has order four, G' is nilpotent. In this paper it is proved that G' is nilpotent if V is non-abelian of order six, but that G' need not be nilpotent for any further groups other than those just mentioned. A side result is that G has nilpotent length at most 2 when V is non-abelian of order pq, p and q primes (non-Fermat, if $|G|$ is even).

A fundamental theorem of Thompson [7] states that if G is a group admitting a fixed free automorphism of prime order, then G is nilpotent. It appears to be well known that if, in this theorem, the group of prime order is replaced by any group of automorphisms of composite order acting in fixed point free manner on G, one can no longer conclude that G is nilpotent. (For the sake of completeness, this fact is proved at the end of § 1.) However, one can frequently draw weaker conclusions concerning G in these cases. For example, D. Gorenstein and I. N. Herstein [4] proved that a group, G, which admits a fixed point free automorphism of order four, has nilpotent length at most two. S. Bauman [1] in 1961 obtained a similar result for the case that the fixed point free operator group was the four-group. Other more general results giving bounds for the nilpotent length of a solvable group, G, admitting various fixed point free operator groups, V, of order prime to $|G|$ can be found in Hoffman [5], Thompson [8] and Shult [6]. In summarizing these results we remark only that the bounds are best possible when V is abelian and subject to a certain restriction on the prime divisors of its order (a restriction which vanishes when $|V|$ and $|G|$ are both odd), but that the bounds are very large otherwise.

In the case that V has order 4, something rather special obtains. Not only does G have nilpotent length 2, but moreover G has a nilpotent commutator subgroup. These findings raise the following question: Let G admit a fixed point free group of operators, V, of order prime to $|G|$. For what groups, V, does this imply nilpotence of the commutator subgroup? From the above-mentioned results of Thompson, Gorenstein

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