PACIFIC JOURNAL OF MATHEMATICS
Vol. 17, No. 2, 1966

SIMPLE n-ASSOCIATIVE RINGS

D. L. OUTCALT

This paper is concerned with certain classes of nonassociative rings. These rings are defined by first extending the associator $(a, b, c) = (ab)c - a(bc)$. The n-associator $(a_1, \cdots, a_n)$ is defined by

$$(1.1)\qquad \begin{aligned} &(a_1, a_2) = a_1 a_2 \,, \\ &(a_1, \cdots, a_n) = \sum_{k=0}^{n-2} (-1)^k (a_1, \cdots, a_k, a_{k+1}a_{k+2}, \cdots, a_n) \,. \end{aligned}$$

A ring is defined to be n-associative if the n-associator vanishes in the ring. It is shown that simple 4-associative and simple 5-associative rings are associative; simple $2k$-associative rings are $(2k - 1)$ associative or have zero center; and simple, commutative n-associative rings, $6 \leqq n \leqq 9$, are associative. The concept of rings which are associative of degree $2k + 1$ is defined, and it is shown that simple, commutative rings which are associative of degree $2k + 1$ are associative. The characteristic of the ring is slightly restricted in all but one of these results.

The concepts of the n-associator and n-associative rings were defined by A. H. Boers [1; Ch. 3 and Ch. 4]. Our results extend Boers' main result that an n-associative division ring is associative with minor restriction on the characteristic [1; Th. 6]. We do not consider 2-associative rings.

To obtain our results, it is necessary to extend the concept of the n-associator. In a ring R, define $S(2j + 1, 2k + 1)$, $1 \leqq j \leqq k$, by defining $S(2j + 1, 2j + 1)$ to be the set of all finite sums of $(2j + 1)$-associators with entries in R, and then by defining $S(2j + 1, 2k + 1)$, $k > j$, to be the set of all finite sums of $(2j + 1)$-associators $(a_1, \cdots, a_{2j+1})$ such that $(a_1, \cdots, a_{2j+1}) \in S(2j + 1, 2k - 1)$ and such that at least one of the $2k - 1$ entries of $(a_1, \cdots, a_{2j+1})$ is in $S(3, 3)$. For example, $(((a_1, a_2, a_3), a_4, a_5), a_6, (a_7, a_8, a_9)) \in S(3, 9)$.

Clearly, a ring R is $(2n + 1)$-associative if and only if $S(2n + 1, 2n + 1) = 0$ in R. This leads us to call a ring $R(2n + 1)$-associative of degree $2k + 1$ if $S(2n + 1, 2k + 1) = 0$ in R. No mention of degree will be made in case $k = n$.

Received November 19, 1964. Presented to the American Mathematical Society, August 28, 1964. This research was supported by United States Air Force Grant No. AFOSR-698-65.