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CHAINS OF MODULES WITH COMPLETELY REDUCIBLE QUOTIENTS

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Consider a left module V over a possibly noncommutative ring R. The objective is to investigate finite or infinite sequences of submodules of V of the form $\{0\} = A_0 \subseteqq A_1 \subseteqq A_2 \cdots$ or of the form $V = A^0 \supseteqq A^1 \supseteqq A^2 \cdots$ where all the quotient modules A_{i+1}/A_i or A^i/A^{i+1} are completely reducible. It is shown that some of the known properties of such series for a module over a ring with minimum condition hold for a more general class of rings, a class which properly includes those satisfying the descending chain condition. The main difficulty which this note has attempted to solve is to generalize these well known theorems from the minimum condition case to a much larger class of rings and modules. The class of rings considered in this note seems to be the natural setting in which to prove these theorems. In spite of the added generality, our proofs are not longer than they would be if the minimum condition were assumed.

All modules considered here will be understood to be left modules. A module V over a ring will be called *simple* provided it contains no proper nonzero submodules and provided also $RV = V \neq \{0\}$. A module is *completely reducible* provided it is a finite or infinite direct algebraic sum of simple modules.

In the next definitions and subsequently, the set inclusion symbol "$\subset$" will always indicate a proper inclusion. The next two definitions are essentially taken from [4, p. 103].

DEFINITION. Suppose V is any left module over an arbitrary ring R. Define $L_0(V)$ to be the zero module. For any positive integer $k = 1, 2, \cdots$ let $L_k(V)$ be defined inductively as the algebraic sum of all submodules Y of V with $L_{k-1}(V) \subset Y$ and with a simple quotient $Y/L_{k-1}(V)$. If $L_\alpha(V)$ has been defined for all $\alpha < \beta$; where β is a limit ordinal, set $L_\beta(V) = \cup\{L_\alpha(V) \mid \alpha < \beta\}$ and define $L_{\beta+1}(V)$ to be the sum of all submodules Y of V with $L_\beta(V) \subset Y$ and with a simple quotient $Y/L_\beta(V)$. The empty sum is taken to be the zero module. The series of submodules $\{0\} = L_0(V) \subseteqq L_1(V) \subseteqq L_2(V) \subseteqq \cdots$ is called the lower Loewy series of V over R.

DEFINITION. For a left module V over a ring R, set $L^0(V) = V$

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