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ON THE NONSINGULARITY OF COMPLEX MATRICES

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Let $A = (a_{ij})$ be a real square matrix of order n with nonnegative entries, and let $M(A)$ be the class of all complex matrices $B = (b_{ij})$ of order n such that, for all i, j, $|b_{ij}| = a_{ij}$. If every matrix in $M(A)$ is nonsingular, we say $M(A)$ is regular, and it is the purpose of this note to investigate conditions under which $M(A)$ is regular.

Many sufficient conditions have been discovered (cf., for instance, [8] and [3], and their bibliographies), motivated by the fact that the negation of these conditions, applied to the matrix $B - \lambda I$, yields information about the location of the characteristic roots. We shall show that a mild generalization of the most famous conditions [2] is not only sufficient but also necessary. (The application of our result to characteristic roots will not be discussed here, but is contained in [5]. See also [7] and [9]).

If

$$(1.1) \qquad a_{ii} > \sum_{i \neq j} a_{ij}\ , \qquad i = 1, \cdots, n\ ,$$

then ([2]) $M(A)$ is regular. Clearly if P is a permutation matrix, and D a diagonal matrix with positive diagonal entries, such that PAD satisfies (1.1), then $M(A)$ is regular. We shall show that, conversely, if $M(A)$ is regular, there exist such matrices P and D so that (1.1) holds.

2. Notation and lemmas.

If $x = (x_1, \cdots, x_n)$ is a vector, x^D is the diagonal matrix whose ith diagonal entry is x_i. If $M = (m_{ij})$ is a matrix, M^v is the vector whose ith coordinate is m_{ii}. A vector $x = (x_1, \cdots, x_n)$ is positive if each $x_j > 0$; x is semi-positive if $x \neq 0$ and each $x_j \geqq 0$. A diagonal matrix D is positive (semipositive) if D^v is positive (semi-positive). If $A = (a_{ij})$ is a matrix with nonnegative entries, a particular entry a_{ij} is said to be dominant in its column if

$$a_{ij} > \sum_{k \neq i} a_{kj}\ .$$

LEMMA 1. *If $e_1, \cdots, e_n$ are nonnegative numbers such that the largest does not exceed the sum of the others, then there exist complex numbers z_i such that*

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