PACIFIC JOURNAL OF MATHEMATICS
Vol. 20, No. 1, 1967

ANOTHER PROOF OF A THEOREM ON RATIONAL CROSS SECTIONS

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The extant proofs of the existence of a rational cross section for a transformation space for a connected solvable linear algebraic group either use a certain amount of algebraic curve theory or restrict themselves to the case of a principal space, where the question is one of galois cohomology, the result being equivalent to the statement that $H^1(G, k) = 0$ for G a k-solvable linear algebraic group. The present proof of the general result may be considered more elementary in that it depends only on the standard facts on fields of rationality of algebraic sets.

The result in question says that *if G is a k-solvable linear algebraic group and V a transformation space for G, all rational over a field k, then there exists a G-invariant dense k-open subset V' of V such that V'/G exists and is rational over k and a cross section k-morphism $V'/G \to V'$ exists.* This statement appears to be somewhat stronger than the original statement [1, Th. 10], but is exactly equivalent to it (except for the purely technical matter of the possible reducibility of V) once one accepts the result that any algebraic transformation space admits a quotient space, provided one restricts to a dense open subset [2].

If V happens to be G-homogeneous the theorem says nothing more than that V has a point rational over k. This special case also implies the general theorem, and that without much labor (in fact the detailed argument in [1] can be much shortened by use of the result quoted at the end of the paragraph above). As for proving the special case just quoted, if one assumes the case $\dim G = 1$ there is a straightforward induction argument on $\dim G$ (again refer for details to [1]) so the crux of the proof is that *a k-homogeneous space for G_a or G_m has a point rational over k.* This note concentrates on the proof of this last statement.

The proof given in [1] that a k-homogeneous space for G_a or G_m has a rational point over k uses some technical information from the theory of algebraic curves, considerably out of the spirit of the present subject. If the homogeneous space happens to be principal homogeneous the question becomes an easy one of galois cohomology [3, pp. 170–171], but some generality has been lost. The present proof is accomplished by starting with a somewhat stronger version of the fact to be proved (Lemma 3 below, which can be taken as the substitute for the Lemma