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OPERATORS AND INNER FUNCTIONS

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Let $L^2_{\mathscr{H}}$ denote the Hilbert space of weakly measurable functions on the unit circle of the complex plane with values in a separable Hilbert space $\mathscr{H}$, and whose pointwise norms are square integrable with respect to Lebesgue measure. We are concerned with invariant subspaces of $L^2_{\mathscr{H}}$, by which we mean closed subspaces invariant under the right shift operator, and will be especially interested in those invariant subspaces which arise from a bounded operator on $\mathscr{H}$, using a construction due to Rota and Lowdenslager. We begin by relating the determinant of the "Rota inner function" of an operator to the characteristic polynomial of the operator, along with a similar interpretation of the minimal polynomial, when $\mathscr{H}$ is finite dimensional. We then consider some general questions about intersections and unions of invariant subspaces, and use the results to establish a factorization theorem for finite dimensional inner functions (the set of all $\mathscr{U}^*\mathscr{V}$, where $\mathscr{U}$, $\mathscr{V}$ are inner, is the same as the set of all $\mathscr{U}\mathscr{V}^*$). We show this theorem false if $\mathscr{H}$ is infinite dimensional, by exhibiting invariant subspaces $\mathscr{M}$, $\mathscr{N}$ (which are also Rota subspaces) such that $\mathscr{M} \cap \mathscr{N} = (0)$—a result of independent interest.

Rota subspaces seem to exhibit all of the pleasant and all of the pathological properties of invariant subspaces in general, and enable one to use properties of operators to provide counterexamples for general questions about invariant subspaces. It is also to be expected that invariant subspaces and their corresponding inner functions, like the analogous theory of characteristic matrix functions [8], can be used to study operators. Our results go in both directions, though they are of more interest, we believe, when using operators to study invariant subspaces.

A word about notation. $H^2_{\mathscr{H}}$ will denote the subspace of $L^2_{\mathscr{H}}$ consisting of all functions with weak analytic extensions to the disk. The inner product of $\mathscr{H}$ will be denoted by (x, y) and the norm of $\mathscr{H}$ by $|x| = (x, x)^{\frac{1}{2}}$. For the inner product in $L^2_{\mathscr{H}}$ we will write

$$[F, G] = \int (F(e^{ix}), G(e^{ix}))d\sigma(x) ,$$

where $d\sigma = (1/2\pi)dx$ is normalized Lebesgue measure on the circle. The norm of $L^2_{\mathscr{H}}$ will be denoted by $\| F \| = [F, F]^{\frac{1}{2}}$. T will always be a bounded operator on $\mathscr{H}$ whose uniform norm $\| T \|$ is less than one. We emphasize that when discussing subspaces of $L^2_{\mathscr{H}}$ the term invariant means invariant under the right shift operator. For the definitions and basic properties of $L^2_{\mathscr{H}}$ and $H^2_{\mathscr{H}}$, consult Helson's book [2].