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# THE EXTENSION OF BILINEAR FUNCTIONALS

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**Using the relationship between bilinear functionals and linear operators we obtain some theorems on the extension of bilinear functionals. To extend bilinear functionals in Hilbert Spaces a special constructional process is given which is a generalization of the usual inner product. This allows the construction of bilinear functionals with special properties. In particular it allows a generalization of the Lax-Milgram Theorem. We also extend the Lax-Milgram Theorem to reflexive Banach Spaces.**

In order to fix the terminology, let $U$ and $V$ be Banach Spaces (real or complex), then by a bilinear functional we mean a function $F$ from $U \times V$ to the complex (or real) numbers such that $F(u, v)$ is linear on $U$ for each fixed $v \in V$ and vice versa. The norm of a bounded bilinear functional $F$, denoted by $\| F \|$, is defined as:

$$\| F \| = \inf \{K > 0 : | F(u, v) | \leqq K \| u \| \| v \| \text{ for all } u \in U, v \in V\} .$$

Hence a bounded bilinear functional is jointly continuous on $U \times V$ in the product topology and we note that:

$$\| F \| = \sup_{\substack{\|x\| \leqq 1 \\ \|y\| \leqq 1}} | F(x, y) | = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} | F(x, y) | \leqq \sup_{\|x\| \, \|y\|=1} | F(x, y) | \leqq \| F \| .$$

If $S$ and $T$ are subspaces of $U$ and $V$ respectively and $B_0$ is a bounded bilinear functional on $S \times T$, then we call $B$ an extension of $B_0$ to $U \times V$ if $B$ is a bounded bilinear functional on $U \times V$ such that $B_0(s, t) = B(s, t)$ on $S \times T$ and $\| B_0 \| = \| B \|$. If such an extension exists we shall say that $B_0$ can be extended to $U \times V$. Furthermore, if each bounded bilinear functional on $S \times T$ can be extended to $U \times V$ we will say that $S$ and $T$ have the bilinear extension property.

## 2. Some extension theorems.

THEOREM 1. *Suppose $U$, $V$, $W$ are normed linear spaces and $S$ and $T$ are subsets of $U$ and $V$ respectively. A necessary and sufficient condition to extend a bounded bilinear operator $B_0$ from $S \times T$ into $W$ to a bounded bilinear operator $B$ from* (Span $S$) $\times$ (Span $T$) *into $W$ is that there exists a constant $c$ such that*

$$\Big\| \sum_{k,j} \alpha_k \beta_j B_0(x_k, y_j) \Big\| \leqq c \Big\| \sum_k \alpha_k x_k \Big\| \, \Big\| \sum_j \beta_j y_j \Big\| .$$