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REMARK ON A PROBLEM OF NIVEN AND ZUCKERMAN

R. T. BUMBY AND E. C. DADE

An integer of an algebraic number field K is called irreducible if it has no proper integer divisors in K. Every integer of K can be written as a product of irreducible integers, usually in many different ways. Various problems have been inspired by this lack of unique factorization. This paper studies the question: When are the irreducible integers of K determined by their norms? Attention is confined to the case in which K is a quadratic field. With this assumption it is possible to give a complete answer in terms of the ideal class group of K and the nature of the units of K.

The fields sought in this problem are those quadratic fields K (with $N: K \to Q$ denoting the norm) which satisfy

Property N: If α is an irreducible integer of K and β is another integer of K such that $N\alpha = N\beta$, then β is also irreducible.

In many cases Property N can be studied by looking at the class group H of K. However the study is complicated by the existence of quadratic number fields K satisfying:

(1) *K is real and $N\varepsilon = +1$, for every unit ε of K.*

When K satisfies (1), we are forced to consider an extended class group H' of K defined as follows:

Two nonzero fractional ideals $\mathfrak{a}, \mathfrak{b}$ are said to be *strongly equivalent* if $\mathfrak{a}\cdot\mathfrak{b}^{-1} = (\gamma)$ is a principal ideal generated by an element γ of positive norm. This is clearly an equivalence relation. The strong equivalence classes form the group H' under the usual multiplication. There are two strong equivalence classes of principal ideals: the class σ consisting of all principal ideals (α) such that one, and hence all, generators of (α) have negative norm; and the identity class 1 of principal ideals (α) all of whose generators have positive norm. Clearly $\sigma^2 = 1$, and the class group H is naturally isomorphic to $H'/\langle\sigma\rangle$.

If K does not satisfy (1), notice that H', as defined above, and the class group H coincide.

In any case, if $\mathfrak{p}$ is any prime ideal of K and $\mathfrak{p}'$ is the conjugate prime ideal, then $\mathfrak{p}\cdot\mathfrak{p}' = (N\mathfrak{p})$. But $N(N\mathfrak{p}) = (N\mathfrak{p})^2 > 0$. So

(2) *$\mathfrak{p}$ and $\mathfrak{p}'$ lie in inverse strong equivalence classes.*

Our main result is