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ERGODIC PROPERTIES OF NONNEGATIVE MATRICES-I

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This paper contains an attempt to develop for discrete semigroups of infinite order matrices with nonnegative elements a simple theory analogous to the Perron-Frobenius theory of finite matrices. It is assumed throughout that the matrix is irreducible, but some consideration is given to the periodic case. The main topics considered are

(i) nonnegative solutions to the inequalities

$$r\sum_k x_k t_{kj} \leqq x_j \qquad (r > 0)$$

(ii) nonnegative solutions to the inequalities

$$r\sum_k x_k t_{kj} \geqq x_j \qquad (r > 0)$$

(iii) the limiting behaviour of sums $P_j(n; r) = \sum_k u_k t_{kj}^{(n)} r^n$ **as** $n \to \infty$, **where** $\{u_k\}$ **is arbitrary nonnegative vector. An extensive use is made of generating function techniques.**

It is well-known that an $n \times n$ matrix with positive elements t_{ij} has an eigenvalue with very special properties: it is positive, greater in modulus than all other eigenvalues, a simple root of the characteristic equation, and associated with unique positive eigenvectors for both the original matrix and its transpose. In the present paper we shall develop some related results when the matrix is infinite (of denumerable order). Although this work was suggested by recent results for Markov chains, we shall not here make the assumption that the matrix is stochastic (i.e. that $\Sigma_j t_{ij} = 1$). Nor shall we place any restrictions on the matrix of the type that it should act as a bounded linear operator on one of the standard sequence spaces. Thus our results are not directly covered by recent extensions of the Perron-Frobenius theorem to linear operators leaving invariant a positive cone in a normed linear space; the relation of our results to these theorems will be discussed in a sequel (part II of the present paper).

It is convenient to relax the requirements that the matrix elements be strictly positive, and to suppose only that they are nonnegative and that the matrix is *irreducible*. If we also assume (as we shall throughout) that the matrix iterates ($T^n = \{t_{ij}^{(n)}\}$) are defined and finite for $n = 2, 3, \cdots$, the condition of irreducibility is equivalent to the condition that for each pair of indices i, j there exists an integer $n > 0$ (depending in general on i and j) such that $t_{ij}^{(n)} > 0$.

As in the Markov chain case the weakening from matrices with