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ON THE RIGIDITY OF SEMI-DIRECT PRODUCTS OF LIE ALGEBRAS

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Roughly speaking, a Lie algebra L is *rigid* if every Lie algebra near L is isomorphic to L. It is known that L is rigid if the Lie algebra cohomology space $H^2(L, L)$ vanishes. In this paper we give an elementary set of necessary and sufficient conditions, independent of Lie algebra cohomology, for the rigidity of a semi-direct product $L = S +_\rho W$, where ρ is an irreducible representation of a semi-simple Lie algebra S on a vector space W. These conditions lead to a number of new examples of rigid Lie algebras. In particular, we obtain a rigid Lie algebra L with $H^2(L, L) \neq 0$.

It follows from [9] that there is only a finite number of isomorphism classes of rigid Lie algebras with a given underlying vector space. The "rigidity theorem" of [9] shows that L is rigid if $H^2(L, L) = 0$. Thus semi-simple Lie algebras are rigid. In general, however, it is difficult to compute $\mathrm{H}^2(L, L)$ and there are few known examples of rigid Lie algebras which are not semi-simple. In considering the rigidity of semi-direct products $L = S +_\rho W$, we avoid the use of Lie algebra cohomology and appeal instead to the "stability theorem" of [10]. Our results essentially reduce the problem of rigidity for such semi-direct products to a classification problem in the theory of semi-simple Lie algebras.

In a series of papers [6] written with an eye towards applications to physics, R. Hermann has obtained results similar to ours in a number of special cases. His method involves a direct computation of $H^2(L, L)$.

1. Preliminaries. Let V be a finite-dimensional real or complex vector space and let $A^2(V)$ denote the vector space of all alternating bilinear maps of $V \times V$ into V. Let $\mathscr{M}$ be the algebraic set in $A^2(V)$ consisting of all Lie algebra multiplications on V. There is a canonical representation of the group $G = GL(V)$ of all vector space automorphisms of V on the vector space $A^2(V)$ defined as follows. If $g \in G$ and $\varphi \in A^2(V)$, then $(g.\varphi)(x, y) = g(\varphi(g^{-1}x, g^{-1}y))$ for all $x, y \in V$. The algebraic set $\mathscr{M}$ is stable under the corresponding action of G on $A^2(V)$. Moreover, the orbits of G on $\mathscr{M}$ correspond precisely to the isomorphism classes of Lie algebra structures on V.

Let $\mu \in \mathscr{M}$ and let $L = (V, \mu)$ be the corresponding Lie algebra. Then L is *rigid* if the orbit $G(\mu)$ is an open subset of $\mathscr{M}$. If V is