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ESTIMATES FOR THE TRANSFINITE DIAMETER WITH APPLICATIONS TO CONFORMAL MAPPING

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Let $f(z)$ be a member of the family S of functions regular and univalent in the open unit disk whose Taylor expansion is of the form: $f(z) = z + a_2z^2 + \cdots$. Let D_w be the image of the unit disk under the mapping: $w = f(z)$. An inequality for the transfinite diameter of n compact sets in the plane $\{T_i\}_1^n$ is established, generalizing a result of Renngli:

$$d(T_1 \cap T_2)\cdot d(T_1 \cup T_2) \leqq d(T_1)\cdot d(T_2)\,.$$

This inequality is applied to derive covering theorems for D_w relative to a class of curves issuing from $w = 0$, arcs on the circle: $|w| = R$ as well as other point sets.

I. Preliminary considerations.

DEFINITION (1.1). Let E be a compact set in the plane. Set:

$$V(z_1, \cdots, z_n) = \prod_{k>l}^{n} (z_k - z_l) \qquad n \geqq 2\,, \quad z_i \in E\,,$$

$$V_n = V_n(E) = \max_{z_1,\cdots,z_n \in E} |V(z_1, \cdots, z_n)|$$

and

$$d_n = d_n(E) = V_n^{2/n(n-1)}\,.$$

The transfinite diameter of E is then defined by: $d = d(E) = \lim_{n\to\infty} d_n$.

A full discussion of the transfinite diameter and related constants can be found in [2, Chapter 7].

The following is a theorem of Hayman [3]:

THEOREM (1.2). *Suppose $f(z)$ is a function meromorphic in the unit disk with a simple pole of residue k at the origin, i.e., the expansion of $f(z)$ about the origin is of the form:*

$$f(z) = \frac{k}{z} + a_0 + a_1 z + \cdots\,.$$

Let D_w denote the image of $|z| < 1$ under the mapping $w = f(z)$ and let E_w denote the complement of D_w in the w-plane. Then: $d(E_w) \leqq k$ with equality if and only if $f(z)$ is univalent.

Using Hayman's theorem is easy to prove the following: