PACIFIC JOURNAL OF MATHEMATICS
Vol. 22, No. 2, 1967

LOCAL ISOMORPHISM OF COMPACT CONNECTED LIE GROUPS

P. F. BAUM

It is shown that two nonisomorphic compact connected Lie groups can be covering groups of each other. This is examined in detail and is related to the question of determining all the compact connected Lie groups belonging to a fixed Lie algebra.

Given a Lie group G it is natural to ask: "What are all the groups locally isomorphic to G?" The answer to this is well known. If $\bar{G}$ denotes the simply connected covering group of G, then any group locally isomorphic to G is obtained by forming the quotient group $\bar{G}/\Gamma$ where Γ is a discrete subgroup of the center of $\bar{G}$. (For this see e.g. [3]). We shall say that a Lie group G_1 covers a Lie group G_2 if there exists a continuous homomorphism of G_1 onto G_2 with discrete kernel. We can then pose the question: "Let G be a Lie group and let $G_1, G_2, \cdots$ be Lie groups such that any group locally isomorphic to G is isomorphic to one and only one of the G_i. Order the set $G_1, G_2, \cdots$ by setting $G_i \geqq G_j$ if G_i covers G_j. How can one describe the structure of this ordered set?" In the case when G is compact connected we give a precise answer to this question.

One surprising result of the investigation is that two compact connected Lie groups can cover each other but not be isomorphic. The simplest example of this is provided by the groups $U(5)$ and $U(5)/\Gamma_2$, where $U(5)$ denotes the 5×5 unitary group and Γ_2 is the subgroup of $U(5)$ consisting of the two matrices $\pm I$. $I =$ identity matrix.

Closely related to this example is the:

PROPOSITION. In $U(n)$ let Γ_k denote the subgroup of all diagonal matrices λI where λ is a complex number such that $\lambda^k = 1$. Then $U(n)/\Gamma_{k_1}$ and $U(n)/\Gamma_{k_2}$ are isomorphic if and only if $k_1 \equiv \pm k_2 \bmod n$.

1. Notation.

If G is a compact connected Lie then $\tilde{G}$ shall denote the covering group of G of the form $H_1 \times H_2 \times \cdots \times H_t \times T^n$ where each H_i is a compact connected simply connected group in one of the Cartan series A_n, B_n, C_n, D_n or is a compact connected simply connected exceptional group, and T^n is the n-dimensional torus group. $T^n = \{(z_1, z_2, \cdots, z_n) \mid z_i \in \boldsymbol{C} \text{ and } |z_i| = 1\}$.

If K is a subgroup of the center of $H_1 \times H_2 \times \cdots \times H_t$ and φ is a homomorphism of K into T^n, then (K, φ) denotes the subgroup of $\tilde{G}$ consisting of all elements $(g, \varphi(g))$ where g ranges over K.