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THE CLOSURE OF THE NUMERICAL RANGE CONTAINS THE SPECTRUM

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The purpose of this paper is to show that the well known theorem in the theory of linear operators in Hilbert space indicated in the title holds for nonlinear operators and to a certain extent for noncontinuous ones, and to provide a constructive method for solving the equations involved.

In different and more precise terms the theorem about to be generalized says:

THEOREM I. *Let T be an everywhere defined linear mapping of a complex Hilbert space $\mathscr{H}$ into itself. Then for any complex number λ at a positive distance $d(\lambda, T)$ from the numerical range of T: $\mathscr{N}(T) = \{(Tx, x), \|x\| = 1\}$, (parentheses indicating scalar product) the equation*

$$\lambda x = Tx - y \tag{1}$$

has a unique solution for every $y \in \mathscr{H}$. The operator $(T - \lambda I)^{-1}$ thus defined is bounded and $\|(T - \lambda I)^{-1}\| \leqq d^{-1}(\lambda, T)$. Moreover, for adequate choices of the averaging factor α depending on T and λ only,

$$(T - \lambda T)^{-1}y = \lim_{n\to\infty} [(1 - \alpha)I + \alpha\lambda^{-1}(T - y)]^n x_0 , \tag{2}$$

where x_0 is any point in $\mathscr{H}$ and $T - y$ the operator mapping x into $Tx - y$.

The theorem having been stated in somewhat more general terms than usual, a proof is needed.

Proof. By definition of $d(\lambda, T)$,

$$|((T - \lambda I)x, x)| = \|x\|^2 \left| \frac{(Tx, x)}{\|x\|^2} - \lambda \right| \geqq d(\lambda, T)\|x\|^2 ,$$

whence it follows by Schwarz' inequality $\|(T - \lambda I)x\| \geqq d(\lambda, T)\|x\|$, proving that $T_\lambda = T - \lambda I$ is a one-one mapping with bounded inverse and $\|T_\lambda^{-1}\| \leqq d^{-1}(\lambda, T)$. By the first inequality above any vector orthogonal to the range $\mathscr{R}(T_\lambda)$ of T_λ must vanish, meaning that $\mathscr{R}(T_\lambda)$ is dense in $\mathscr{H}$. Thus for any $y \in \mathscr{H}$ there is a sequence $\{x_n\}$ such that $T_\lambda x_n \to y$; since T_λ^{-1} is bounded $\{x_n\}$ converges to some element x. Setting $x_n^* = x_n - x$ and $y^* = y - T_\lambda x$, one obtains from the