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ADDITION THEOREMS FOR SETS OF INTEGERS

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Let C be a set of integers. Two subsets A and B of C are said to be complementing subsets of C in case every $c \in C$ is uniquely represented in the sum

$$C = A + B = \{x \mid x = a + b, a \in A, b \in B\} .$$

In this paper we characterize all pairs A, B of complementing subsets of

$$N_n = \{0, 1, \cdots, n-1\}$$

for every positive integer n and show some interesting connections between these pairs and pairs of complementing subsets of the set N of all nonnegative integers and the set I of all integers. We also show that the number $C(n)$ of complementing subsets of N_n is the same as the number of ordered nontrivial factorizations of n and that

$$2C(n) = \sum_{d \mid n} C(d) .$$

The structure of complementing pairs A and B has been studied by de Bruijn [1], [2], [3] for the cases $C = I$ and $C = N$ and by A. M. Vaidya [7] who reproduced a fundamental result of de Bruijn for the latter case. In case $C = N$ it is easy to see that $A \cap B = \{0\}$ and that $1 \in A \cup B$. Moreover, if we agree that $1 \in A$, it follows from the work of de Bruijn, that, except in the trivial case $A = N$, $B = \{0\}$, A and B are infinite complementing subsets of N if and only if there exists an infinite sequence of integers $\{m_i\}_{i \geqq 1}$ with $m_i \geqq 2$ for all i, such that A and B are the sets of all finite sums of the form

$$(1) \qquad \begin{aligned} a &= \sum x_{2i} M_{2i} , \\ b &= \sum x_{2i+1} M_{2i+1} \end{aligned}$$

respectively where $0 \leqq x_i < m_{i+1}$ for $i \geqq 0$ and where $M_0 = 1$ and $M_i = \prod_{j=1}^{i} m_j$ for $i \geqq 1$. In the remaining case, when just one of A and B is infinite, the same result holds except that the sequence $\{m_i\}$ is of finite length r and that $x_r \geqq 0$. Similar results can also be obtained in the case of complementing k-tuples of subsets of N for $k > 2$.

The case $C = I$ is much more difficult and, while sufficient conditions are easily given, necessary and sufficient conditions that a pair A, B be complementing subsets of I are not known. As an example of sufficient conditions, we note that if A and B are as in (1) above, then A and $-B$ form a pair of complementing subsets of I. This is