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AW^*-ALGEBRAS ARE QW^*-ALGEBRAS

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G. A. Reid has introduced a class of B^*-algebras called QW^*-algebras which includes the W^*-algebras and which is included in the class of AW^*-algebras. In this paper it is shown that the QW^*-algebras are exactly the AW^*-algebras.

We shall use Reid's notation (see [2]) without further explanation.

THEOREM. *Let A be an AW^*-algebra. Then A is a QW^*-algebra.*

Proof. Let B be a norm-closed *-subalgebra of A. Then, using [1; Theorem 2.3] we see that A contains a hermitian idempotent P such that PA is the right annihilator of B. Since B is a *-subalgebra we see that the left annihilator of B is $(PA)^* = AP$. Thus $B_0 = AP \cap PA = PAP$ and $B_{00} = (I - P)A(I - P)$. It follows by [1; Theorem 2.4] that $B_{00} \supset B$ is an AW^*-algebra with identity $I - P$.

Using the Gelfand-Naimark Theorem [3; 244] we can consider B_{00} as an algebra of operators on a hilbert space H where the identity in B_{00} corresponds to the identity operator I_H in H. Let $(\mathscr{T}, \mathscr{S})$ be a double centraliser on B and let T be the element of $\mathscr{B}(H)$ corresponding to $(\mathscr{T}, \mathscr{S})$ under the isomorphism in [2; Proposition 3]. We have $TB \subset B$, $BT \subset B$ and wish to show that there is an element S of B_{00} with $SL = TL$ and $LS = LT$ for all $L \in B$.

We may clearly suppose T to be symmetric since the general case follows by considering separately the real and imaginary parts of T. Let K be the closed linear subspace of H generated by BH and P_K the orthogonal projection onto K. Let $\{F_\lambda\}$ be the spectral family of T [4; p. 275] and put $E_\lambda = P_K F_\lambda = F_\lambda P_K$. $\{E_\lambda\}$ is essentially the spectral family of T considered as an operator in K. Define

$$C_\lambda = \{P_K f(T); f \in C(\sigma(T)), f(\lambda') = 0 \quad \text{for } \lambda' \leqq \lambda\}$$
$$D_\lambda = \{P_K f(T); f \in C(\sigma(T)), f(\lambda') = 0 \quad \text{for } \lambda' \geqq \lambda\}$$

where $C(\sigma(T))$ is the set of continuous complex valued functions on $\sigma(T)$. The elements of C_λ, D_λ are essentially functions of T in $\mathscr{B}(K)$. Since the elements of C_λ and D_λ are limits in the uniform operator topology of sequences of polynomials in T we see that $C_\lambda B, D_\lambda B, BC_\lambda$ and BD_λ are subsets of B and hence of B_{00}. Using Kaplansky's result [1; Theorem 2.3] we can find an orthogonal projection $P_\lambda \in B_{00}$ such that $P_\lambda B_{00}$ is the right annihilator of BC_λ in B_{00}. Since B_{00} contains I_H we see $P_\lambda \in P_\lambda B_{00}$ and so $BC_\lambda P_\lambda = \{0\}$ and $P_\lambda C_\lambda B = \{0\}$. Thus for $\xi \in BH$, and hence for $\xi \in K$, $P_\lambda C_\lambda \xi = \{0\}$. However for $\xi \in H \ominus K$, $C_\lambda \xi = \{0\}$ and